DISCOVERING THE ART OF MATHEMATICS

MUSIC

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

Christine von Renesse,
with Volker Ecke, Julian F. Fleron and Philip K. Hotchkiss

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(Rev.: 2015-09-04)

Acknowledgements

This work has been supported by the National Science Foundation under awards NSF0836943 and NSF1229515. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

These materials are also based on work supported by Project PRIME which was made possible by a generous gift from Mr. Harry Lucas.

These materials were originally developed for use in the course Mathematical Explorations at Westfield State. They are now part of the curriculum library for Discovering the Art of Mathematics, all of whose volumes are freely available at http://artofmathematics.org .

The Department of Mathematics at Westfield State has provided fertile ground and enthusiastic support for pedagogical innovation, scholarship on teaching and learning in mathematics, and classroom experimentation. Under the direction of Chairs Karin Vorwerk and previously John Judge, and the organization of Deb Samwell, the department continues to be deeply committed to student learning - including in its many required core courses. We are thankful to all departmental faculty, current and past, for their support of Discovering the Art of Mathematics. We are grateful for the support of many at our institution over many years. In particular we thank those in Academic Affairs, our fine library faculty, Robert Martin, William Lopes, Louann D'Angelo, Elizabeth Preston, and Andrew Bonacci.

We are grateful for the support our our Advisory Board: Jere Confrey, David Farmer, Sandra Laursen, Carmel Schettino, Deborah Schifter and Dorothy Wallace. We thank Lee Zia for his guidance and encouragement.

Student interns Jillian Bates, Chelsea Baker, Joey Grabowski, Jen Roy, Rachel Cloutier, Zach Lancto, Alyssa Danilow, and Barbara Mulvaney helped with many tasks that supported the project.

Thanks to Sage Fox Consulting, Ken Rath and Emily DeHaro-Otero in particular, for their work on assessment and evaluation.

A great debt of gratitude to Tom Hoogendyk who designed and keeps running our Discovering the Art of Mathematics website.

The author would like to thank both her immediate and extended family for continual inspiration, support, and love. She would also like to thank her colleagues on the Westfield State campus for mathematical guidance and inspiration.

Contents

Preface

This book is a very different type of mathematics textbook. Because of this, users new to it, and its companion books that form the Discovering the Art of Mathematics library[1], need context for the book's purpose and what it will ask of those that use it. This preface sets this context, addressing first the Explorers (students), then both Explorers and Guides (teachers) and finishing with important information for the Guides.

0.1 Notes to the Explorer

"Explorer?"

Yes, that's you - an Explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a book intended to be read like a traditional book. This book is really a guide. It is a map. It is a route of trail markers along a path through part of the vast world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore - to take a surprising, exciting, and beautiful journey along a meandering path through a great mathematical continent.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by exercises that mimic examples laid out for you to ape. Rather, the majority of each chapter is made up of Investigations. Each chapter has an introduction as well as brief surveys and narratives as accompaniment, but the Investigations form the heart of this book. They are landmarks for your expedition. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover mathematics. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking and mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is being discovered now than at any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! Fundamental questions in mathematics - some hundreds of years old and others with \$ 1 Million prizes - are

[1] All available freely online at `http://artofmathematics.org/books`.

being solved. In the time period between when these words were written and when you read them important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning arithmetical and quantitative skills, statistical reasoning, and applications of mathematics. These are important, to be sure. But there is more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is perhaps its most powerful, driving force. As the famous **Henri Poincaré** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as a liberal art and as a science. As a powerful science, it shapes our technological society and serves as an indispensable tool and as a language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in other fine, accessible books. Instead, our purpose is to journey down a path that values mathematics for its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was preserved in monasteries. The classical **liberal arts** organized knowledge in two components: the ***quadrivium*** (arithmetic, music, geometry, and astronomy) and the ***trivium*** (grammar, logic, and rhetoric) which were united by philosophy. Notice the central role of mathematics in both components. During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts except in contemporary classrooms and textbooks where the focus of attention has shifted solely to its utilitarian aspects. If you are a student of the liberal arts or if you want to study mathematics for its own sake, you should feel more at home on this expedition than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more!

In your exploration here you will see that mathematics is a human endeavor with its own rich history of struggle and accomplishment. You will see many of the other arts in non-trivial roles: art, music, dance and literature. There is also philosophy and history. Students in the humanities and social sciences, you should feel at home here too. There are places in mathematics for anyone to explore, no matter their area of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

We hope that your discoveries and explorations along this mathematical path will help you glimpse some of this beauty. And we hope they will help you appreciate Russell's claim:

> ... The true spirit of delight, the exultation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Finally, we hope that your discoveries and explorations enable you to make mathematics a part of your lifelong educational journey. For, in Russell's words once again:

> ... What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have beaconed people to explore the many continents of mathematics throughout humankind's history.

0.2 Navigating This Book

Intrepid Explorer, as you ready to begin your journey, it may be helpful for us to briefly describe basic customs used throughout this book.

As noted in the Preface, the central focus of this book is the **Investigations**. They are the sequences of problems that will help guide you on your active exploration of mathematics. In each chapter the Investigations are numbered sequentially in bold. Your role will be to work on these Investigation individually or cooperatively in groups, to consider them as part of homework assignments, to consider solutions to selected Investigations that are modeled by your fellow explorers - peers or your teacher - but always with you in an active role.

If you are stuck on an Investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us:

> If there is no struggle, there is no progress.

Or what **Shelia Tobias** (American mathematics educator; 1935 -) tells us:

> There's a difference between not knowing and not knowing *yet*.

Keep thinking about the problem at hand, or let it ruminate a bit in your subconscious, think about it a different way, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Independent Investigations are so-called to point out that the task is more involved than the typical Investigations. They may require more significant mathematical epiphanies, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The **Connections** sections are meant to provide illustrations of the important connections between the mathematics you're exploring and other fields - especially in the liberal arts. Whether you complete a few of the Connections of your choice, all of the Connections in each section, or are asked to find your own Connections is up to your teacher. We hope that these Connections sections will help you see how rich mathematics' connections are to the liberal arts, the fine arts, culture, and the human experience.

Further Investigations, when included, are meant to continue the Investigations of the mathematical territory but with trails to significantly higher ground. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory - you are bushwhacking on less well-traveled trails.

In mathematics, proof plays an essential role. Proof is the arbiter for establishing truth and should be a central aspect of the sense-making at the heart of your exploration. Proof is reliant on logical deductions from agreed upon definitions and axioms. However, different contexts suggest different degrees of formality. In this book we use the following conventions regarding **definitions**:

- An *Undefined Term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.
- An ***Informal Definition*** is italicized and bold- faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.
- A **Formal Definition** is bolded the first time it is used. This is a formal definition that is suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a **Biographical Name** appears it is bolded and basic biographical information is included parenthetically to provide historical, cultural, and human connections.

In mapping out trails for your explorations of this fine mathematical continent we have tried to uphold the adage of **George Bernard Shaw** (Irish playwright and essayist; 1856 - 1950):

> I am not a teacher: only a fellow-traveler of whom you asked the way. I pointed ahead – ahead of myself as well as you.

We wish you wonderful explorations. May you make great discoveries, well beyond those we could imagine.

0.3 Directions for the Guides

Faithful Guide, you have already discovered great surprise, beauty and excitement in mathematics. This is why you are here. You are embarking on a wonderful journey with many explorers looking to you for bearings. You're being asked to lead, but in a way that seems new to many.

We believe telling is not teaching. Please don't tell them. Answer their questions with questions. They may protest, thinking that listening is learning. But we believe it is not.

This textbook is very different from typical mathematics textbooks in terms of structure (only questions, no explanations) and also of expectations it places on the students. They will likely protest, "We're supposed to figure this out? But you haven't explained anything yet!" It is important to communicate this shift in expectations to the students and explain some of the reasons. That's why we have written the earlier sections of this preface, which can help do the explaining for us (and for you).

You need support as well. A shift in pedagogy to a more inquiry-based approach may be subtle for some, but for many it is a great leap. Understanding this we have assembled an online resource to support teachers in the creation and nurturing of successful inquiry-based mathematics classrooms. Available online at `http://artofmathematics.org/classroom` it contains a wealth of information - in many different forms including text, data, videos, sample student work - on many critical topics:

- Why inquiry-based learning?
- How to get started using our books…
- A culture of curiosity
- Learning contracts
- Grouping students
- Choosing materials - Mixing It Up
- Asking good questions
- Creating inquiry-based activities
- Making mistakes
- Cool things
- Proof as sense-making
- Homework stories
- Exams
- Posters
- Assessment: Student Solution Sets
- Evaluating the effectiveness of inquiry-based learning
- … and much more …

We wrote the books that make up the Discovering the Art of Mathematics library because they have helped us have the most extraordinary experiences exploring mathematics with students who thought they hated mathematics and had been disenfranchised from the mathematical experience by their past experiences. We are encouraged that others have had similar experiences with these materials. We love to hear success stories and are also interested in hearing about things that might need to be changed or did not work so well. Please feel free to share your stories and suggestions with us: `http://artofmathematics.org/contact`.

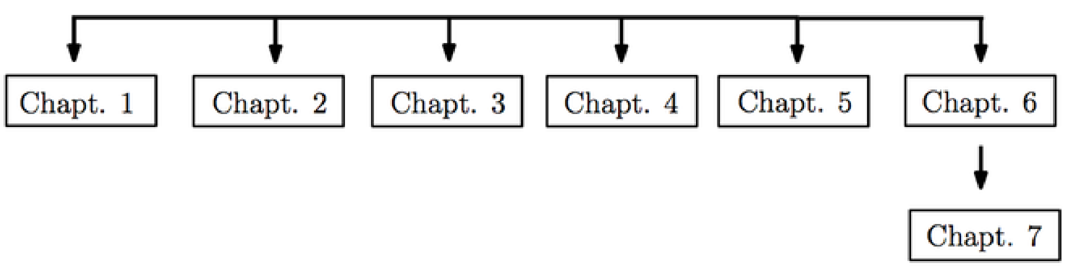
0.3.1 Chapter Dependencies

Guides are encouraged to pick and choose topics freely, from this book and others in the Discovering the Art of Mathematics series, depending on their interests and those of their students. The chapter dependencies in this book are as follows:

Discovering the Art of Mathematics: Music

Chapter Dependencies

The first level are the chapters that can be used independently. The arrows emanating from these chapters indicate which of the remaining chapters depend on them.



Chapter 1

A little Introduction

1.1 Mathematics, Music and Art

Mathematics is music for the mind; music is mathematics for the soul.

Anonymous

This is a not a regular textbook. This is a book which makes you think and write and discuss. I hope you read the "Notes to the explorer" preface.

Before we start diving into a topic, we want to think about the connection of mathematics and the arts.

1. What is mathematics? Find a good definition.

2. What is music? Find a good definition.

3. What is art? Do you think mathematics is an art? Why or why not?

4. **Classroom Discussion:** Compare your definitions with your classmates and your professor and agree on definitions for mathematics, music, dance and art.

Mathematics is everywhere in art, in particular in music even if the artists are not aware of it. This book will show many different areas of music that are built on concepts of mathematics. Discovering the mathematics will deepen your appreciation – not only of the mathematics but also of the artform itself.

5. How often do you listen to music?

6. Why do you think music is so important for you? For humans in general?

7. What is your favorite piece of music? Why?

8. Analyze your piece of music: can you find any structure? Consider rhythm, melody, general format, chords, lyrics, loudness, ...

9. **Classroom Discussion:** Share the structure that you found in your piece with your classmates. Are there any common structures? Any common themes? Do you see any mathematics yet?

1.2 Some Standard Music Notation

You can skip this section for the first read, just know that it is there in case you need it.

On the staff, see Figure 1.1, each line and each space between lines corresponds to one of the note names:

$$C, D, E, F, G, A, B, C, \cdots .$$

The top part of the staff is for the right hand of the pianist while the bottom staff is for the left

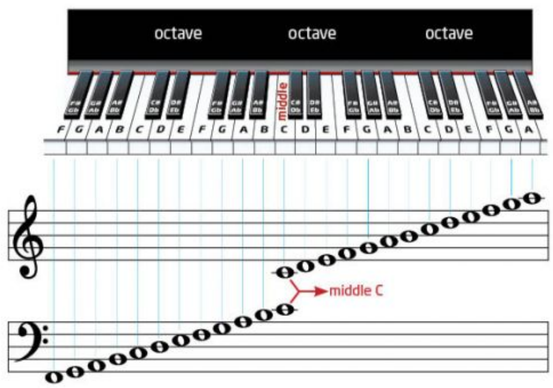


Figure 1.1: Standard Music Notation

hand. The notation for both hands is different, since the keys you play with your left hand have lower sounds than the ones you play with your right hand! Now the notes on the staff in Figure 1.1 are only for the white keys of the piano. Here is how you notate the black keys: If you want a note for the key to the top right of F, which is called F ***sharp*** you write a $\sharp$ in front of the note. See Figure 1.2. If you want a note for the key to the top left of, say E, which is called E ***flat***, you write a $\flat$ in front of the note. The $\sharp$ and the $\flat$ are called ***accidentals***. In this way many keys

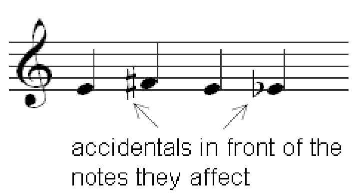


Figure 1.2: Accidentals

have two names, e.g. $F\sharp = G\flat$. You can find details about the history and details of tuning in Chapter 4.

Chapter 2

Exploration of Rhythms and Pascal's Triangle

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made of ideas. His patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way.
>
> **G.H. Hardy** (English Mathematician; 1877 - 1947)

2.1 Counting Rhythms

In this section we will explore rhythms from a mathematical point of view. A rhythm can provide structure to a musical piece but it is also possible that a piece of music consist only of rhythms played, for instance, by drums. There are many interesting questions we could ask about the structure of rhythms and how to combine several of them, but this section will focus on "how many rhythms there are". To begin our inquiry we need to be able to notate rhythms and develop a common language.

1. Listen to the rhythm *Son* at `https://www.youtube.com/watch?v=SnjqxgtLJlA` and `https://www.youtube.com/watch?v=UG9NacR29zM`. Can you notate the rhythm somehow? You can use the standard musical notation, but find at least one other way of representing the rhythm. Describe the rhythm to someone else on paper such that they would be able to reproduce the rhythm after reading your description.

2. **Classroom Discussion:** Compare the different ways of notations for rhythms. Which one seems the best to you? Why?

3. **Classroom Discussion:** Discuss how finding a notation for a rhythm is connected with mathematics? Which aspects of mathematics did you learn about?

The rhythm you just heard is called Son or 3/2 clave. It is the basic rhythm of salsa music and as such known everywhere in the world. We will use a binary notation for rhythms, where we write a 1 for a note (when we play a note or clap) and a 0 for rest. In this notation the first part of the Son rhythm looks like 10010010. Why is this rhythm so special? According to Godfried Toussaint [13], Professor at McGill University, Montreal, "it is one of the most famous rhythms in the world. In Cuba it goes by the name tresillo and in the USA it is often called Habanera. It is also found widely in West African traditional music." Toussaint has done a lot of work on comparing rhythms and looking at them from a geometric point of view.

4. Independent Investigation: The first part of Son consists of 3 notes and 5 rests on a total of 8 counts. We first would like to know how many possible rhythms there are given 3 notes (and 5 rests) on 8 counts. This will be our first mathematical exploration. Work in groups and take your time. Document your work, reason why attempts worked or didn't work. Consider for instance the patterns if you have 1,2 or 3 notes on 3 counts, and 1,2,3 or 4 notes on 4 counts. Have fun!

5. Classroom Discussion: Compare your results from the independent investigation: how many rhythms are possible given 8 counts and 5 notes?

There are many different ways to approach this problem and you found at least one of them. The next questions will help you to find other strategies and connect the different approaches with each other.

6. Given 2 counts, how many ways are there to have a rhythm with 1 note?

7. Given 2 counts, how many ways are there to have a rhythm with 2 notes?

8. Given 3 counts, how many ways are there to have a rhythm with 1 note?

9. Given 3 counts, how many ways are there to have a rhythm with 2 notes?

10. Given 3 counts, how many ways are there to have a rhythm with 3 notes?

11. Given 4 counts, how many ways are there to have a rhythm with 1 note?

12. Given 4 counts, how many ways are there to have a rhythm with 2 notes?

13. Given 4 counts, how many ways are there to have a rhythm with 3 notes?

14. Given 4 counts, how many ways are there to have a rhythm with 4 notes?

15. Can you see a pattern in the above results? Try playing with the numbers. Fill in the next ones using the pattern you found and then check them.

16. Given 5 counts, how many ways are there to have a rhythm with 1 note?

17. Given 5 counts, how many ways are there to have a rhythm 2 notess?

18. Given 5 counts, how many ways are there to have a rhythm 3 notes?

19. Given 5 counts, how many ways are there to have a rhythm 4 notes?

20. Given 5 counts, how many ways are there to have a rhythm 5 notes?

> **21.** Independent Investigation: Take your notebook and write the above numbers in a triangle (pyramid), the rows corresponding to the number of counts and the "columns" corresponding to the number of notes. Can you see a pattern now? Describe the symmetries you observe. Think about what happens if you have no notes at all – how many rhythms are there?

2.2 Pascal's Triangle

The pattern that emerges is called ***Pascal's triangle***. It becomes a full symmetric triangle when you add on the left side of the triangle the following numbers:

22. Given 2 counts, how many ways are there to have a rhythm with 0 notes?

23. Given 3 counts, how many ways are there to have a rhythm with 0 notes?

24. Given 4 counts, how many ways are there to have a rhythm with 0 notes?

25. Given 5 counts, how many ways are there to have a rhythm with 0 notes?

Row Number = Number of Counts

Column Number = Number of Notes

Numbers start at zero.

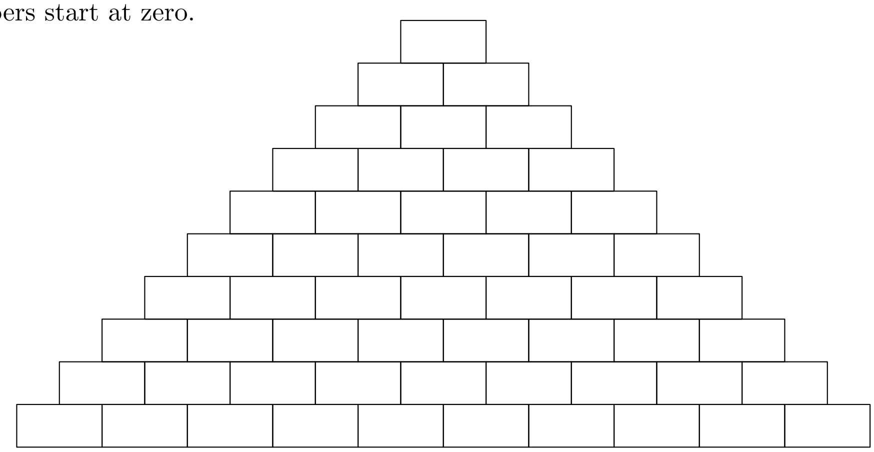


Figure 2.1: Number of Ways to have Rhythms.

Now, given one row of the triangle, how do you find the next row without having to count out the number of rhythms? You can fill your numbers into the empty triangle in Figure 2.1 to help you find the pattern.

26. Classroom Discussion: What are the patterns you find in Pascal's triangle? How can you use patterns to find the next rows without having to count rhythms?

The triangle was studied by **Blaise Pascal** (French Mathematician; 1623 - 1662), although it had been described centuries earlier by the **Yanghui** (Chinese Mathematician; 1238 - 1298) and **Omar Khayyam** (Persian Astronomer and Poet; 1048 - 1131). It is therefore known as the Yanghui triangle in China. See Figure 2.2 for pictures of Pascal, Yanghui and Khayam.

Figure 2.2:

Back to our question: how many rhythms there are for 8 counts and 3 notes. If you ask mathematicians, they might suggest to compute

$$\binom{8}{3} = \frac{8!}{3!5!} \tag{2.1}$$ [1]

Here the exclamation mark stands for *factorial*, which you compute in the following way: $8! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots 8$.

Wow, that looks really complicated! Lets try to understand this for our example. If we have 8 counts and 3 notes then we have to compute $\binom{8}{3} = \frac{8!}{3!5!} = \frac{6 \cdot 7 \cdot 8}{3 \cdot 2 \cdot 1} = 56$. So the mathematician claims there are 56 different rhythms. Does 56 agree with your answer from before? The following investigations will help us understand why this computation works.

[1] Mathematicians write for entry $(k+1)$ in row $(n+1)$ in Pascal's triangle: $\binom{n}{k}$ and say n *choose* k. They have a general equation to compute the entries:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Mathematicians don't usually think about notes and rhythms, for them n choose k computes the number of possibilities to choose sets of k objects out of n objects. But we can think of choosing k positions for notes out of 8 available counts.

Since we have 8 counts, we have 8 places where we can place the 3 notes. If a problem seems too hard, mathematicians like to make it easier. For our example, we will ignore for a moment that the second note should be played *after* the first note.

27. In how many places can you put the first note?

28. After choosing your first note, how many places are left for the second note?

29. After choosing the second note, how many places are left for the third note?

30. How can you use these numbers to find the number of possibilities to place all 3 notes?

We have to be careful, because if you count this way your first note might actually sound after the second note, and that would not be appropriate. From all the ways we can order the 3 notes we only want to consider *one*.

31. In how many ways can you order 3 notes? Imagine you had 3 flower pots and wanted to know in how many different ways you could arrange them in a line...

32. Explain how the expression $\frac{6 \cdot 7 \cdot 8}{3 \cdot 2 \cdot 1}$ relates to Investigation **31**.

33. **Classroom Discussion:** What is a *proof* in mathematics? What is a *conjecture*? Can we use examples to be sure that a conjecture is true?

34. Prove that we can use the factorial equation (2.1) to compute the number of possible rhythms.

There are so many methods to solve our problem of finding the number of possible rhythms with 3 notes in 8 counts. Maybe you did it in one of the above ways? Lets try another method:

If you choose the first note on the first count, here is the list of all the possibilities to choose the other two notes:

1 2 3	1 3 4	1 4 5	1 5 6	1 6 7	1 7 8
1 2 4	1 3 5	1 4 6	1 5 7	1 6 8	
1 2 5	1 3 6	1 4 7	1 5 8		
1 2 6	1 3 7	1 4 8			
1 2 7	1 3 8				
1 2 8					

There are $6 + 5 + 4 + 3 + 2 + 1 = 21$ possibilities for this *block* of numbers.

35. Create a list as above for starting the first note on the second count. How many possibilities are there?

36. Continue creating lists and counting the possibilities.

37. For every block of numbers you found a sum (of the number of possibilities, e.g. the 21 above), can you find all these numbers in Pascals triangle? Circle them in your triangle. Now find our result 56 in the triangle and circle it, too. Describe what you see.

38. The pattern you see is also called the hockey stick pattern. Can you see why? Does the same pattern work for other numbers in the triangle (i.e. when you *move* the hockey stick)?

2.3 Further Investigation

F1. There are many more exciting patterns in Pascals triangle! Find at least three.

One of the patterns you probably found is the addition pattern: when adding two adjacent numbers the result will be right beneath the two numbers that you added. Mathematicians love finding patterns but they also wonder about why the patterns occur and how you can be sure that they will continue to happen. Our goal now is to understand *why* the addition pattern in Pascal's triangle occurs and to make sure that it will still happen for many counts.

F2. In Figure 2.3, fill all possible rhythms in the respective boxes.

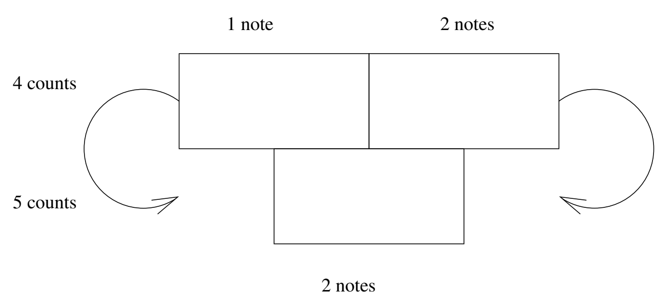


Figure 2.3: Part of Pascal's Triangle.

F3. Now look at the rhythms you filled into Figure 2.3, can you see any structure that suggests how the rhythms in the upper boxes are connected to the rhythms in the box below? Explain the structure you found.

F4. Go to another place in Pascal's triangle and choose three similarly positioned boxes. Fill them with rhythms and see if your structure applies here as well.

F5. Does your structure apply to the top of the triangle?

F6. Explain in your own words why the addition pattern in Pascal's triangle occurs, using the structure you found. Be specific in your arguments.

F7. Explain why the hockey stick pattern in Pascal's triangle is always true.

F8. Read the paper "The Rascal Triangle" (`http://www.maa.org/sites/default/files/pdf/pubs/cmj393-395.pdf`) written by middle school students Alif Anggoro, Eddy Liu and Angus Tulloch. Does this influence or change your thinking about mathematics research?

2.4 Interlude: Clapping Music

You explored one mathematical aspect of rhythms in the last section, which was mostly based on counting. Mathematicians call this kind of mathematics ***Combinatorics***. Now we want to see some musical application of rhythm structures by playing a piece composed by **Steve Reich** (American Composer; 1938 -).

Practice the following rhythm: 111011010110. One person keeps clapping this rhythm. We want to cycle through the rhythm by moving one count from the beginning to the end. For instance for the first time, we move the first note to the end and start with the second note 110110101101. A second person claps the cycled rhythm starting at the same count as the first player starts his or her rhythm. After listening to this combination the second person moves yet one more count over and is now clapping 101101011011. Keep going until both players clap the same rhythm again. You can listen to the piece at `https://www.youtube.com/watch?v=lzkOFJMI5i8`.

39. How do you like the resulting piece of music? What is appealing to you? What do you dislike?

Steve Reich was born in New York, on October 3, 1936, and is currently living in Manhattan. After studying philosophy he turned to music and explored many different techniques of composition. His style is labeled "minimalist music". He does for instance play the same piece on two different instruments but using different tempo (this is called **phasing**). He also used **tape loops**, recording rhythms on tape and then playing them back over and over again in either the same or different tempo. Recently the New York Times called him "our greatest living composer". In April 2009 Steve Reich was awarded the Pulitzer prize in Music for his composition 'Double Sextet'.

Figure 2.4: **Evelyn Glennie** (Scottish Percussionist; 1965 -) opens her recital at the Ormond Beach Performing Arts Center playing Clapping Music by Steven Reich on the wooden blocks. Glennie is the worlds first full-time solo percussionist. She is profoundly deaf.

Scottish percussionist Evelyn Glennie opens her recital at the Ormond Beach Performing Arts

Center playing Clapping Music by Steven Reich on the wooden blocks. Glennie is the worlds first full-time solo percussionist. She is profoundly deaf.

40. How many times does the rhythm 111011010110 need to be shifted in order to sound like the first rhythm again? Find a way to explain your answer that doesn't require listening to the piece.

41. Draw a picture or diagram to support your reasoning in investigation 40.

42. Find a rhythm on 8 counts that will sound the same after being shifted *exactly* 4 times. It should not sound the same after any other shift less than 4. Explain your strategy.

43. Can you find several rhythms for investigation 42? Explain why or why not.

44. Find a rhythm on 8 counts that will sound the same after being shifted *exactly* 5 times. It should not sound the same after any other shift less than 5. Explain your strategy.

45. Can you find several rhythms for investigation 44? Explain why or why not.

46. Can you find a general answer for investigations 42 and 44? The question is: Are there any rhythms on n counts that sound the same after being shifted *exactly* m times? It should not sound the same after any other shift *less than* m times.

47. It turns out that we can ask the question in a slightly different way, maybe you did already come across this problem. The question is now: Are there any rhythms on n counts that sound the same after being shifted m times? It can also sound the same at other shifts less than m times.

48. Compose your own piece of music/rhythm using the above ideas. You can use the software *ABC drums* at `http://www.sju.edu/~rhall/Multi/drums.html` to help you play and record your composition.

2.5 What is mathematics?

Now that you have done some investigations, let's think about what mathematics is. How do these investigations compare to the mathematics you have seen in high school? Is it harder oder easier? Do you like it better or not?

49. Read Lockhart's Lament `http://www.maa.org/devlin/LockhartsLament.pdf` and write a response to it addressing the above questions.

50. Do you think mathematics is an art? Why or why not?

51. **Classroom Discussion:** Share your thoughts about Lockhart's lament with your classmates. Did the reading change your perception of this mathematics class?

2.6 Further Investigations

There is much more that can be done with clapping music, this was just a little taste of it. The following investigation gives you motivations and ideas to go further into the topic and do your own projects.

F9. Given 12 counts how many rhythms are there that lead to interesting pieces of clapping music, similar to Reich's piece? You can read the paper "Clapping Music - a Combinatorial Problem" by Joel Haack, published in the College Mathematics Journal (available online). This is a bigger mathematical challenge, you will have to learn about groups, permutations, and some combinatorics to understand the paper!

Evelyn Glannie has a wonderful Ted talk you should listen too: `http://www.ted.com/talks/evelyn_glennie_shows_how_to_listen.html`. The following investigations can help you focus on different aspects of the talk, but there is much more to take away from it.

F10. How is Evelyn's idea of truly listening to someone (and not being judgmental) relevant in your class? In your life?

F11. Is there a "right" and "wrong" in music? in mathematics?

F12. Do you need to be creative in music? in mathematics?

F13. Evelyn says that you need to experiment with a drum before you can start to make music with it. How is this related to doing mathematics?

F14. In music you can practice playing an instrument or you can interpret music in your own way. Can you find equivalent aspect in doing mathematics?

F15. Evelyn says that she wonders "why she is practicing music". She "needs to have a reason". Do you feel like this when you are doing mathematics?

Chapter 3

Understanding Rhythm-Palindromes

> Music is the pleasure the human mind experiences from counting without being aware that it is counting.
>
> **Gottfried Leibniz** (German Mathematician and Philosopher; 1646 - 1716)

3.1 Palindromes

In Chapter 2 we looked at the combinatorics of rhythms in general. In this chapter we want to look more closely at some especially beautiful rhythms: palindromes. But before we start we need to talk about how to notate rhythms. There are many different ways we can do this; for this chapter we want to focus on two of them.

1. Binary notation, for instance 10110110, with 1s as notes and 0s as rests.

2. Geometric notation, where we notate the counts around a circle and mark the notes as dots. If you connect the dots you can use the geometric shape, called a ***polygon***, to visualize the rhythm. See Figure 3.1.

We want to find and count rhythms that are *palindromes*. You may be familiar with the notion of a palindrome from words like ANNA which read the same forwards and backwards.

Many composer have used palindromes in their compositions. In Figure 3.3 you can see an excerpt from Alban Berg's opera "Lulu" that shows a rythmical palindrome. Haydn's 47th synphony is sometimes called "The Palindrome" because of his use of melodic and rhythmical palindromes, see Figure 3.2 (Second part of the Minuet).

Godfried Toussaint [13] describes a **palindrome** as a rhythm that sounds the same if you play it forward or backward in the circle notation. Figure 3.1 shows a palindrome in that sense: If you play to the right, starting at the top, you get 10110110, if you play to the left you get 10110110.

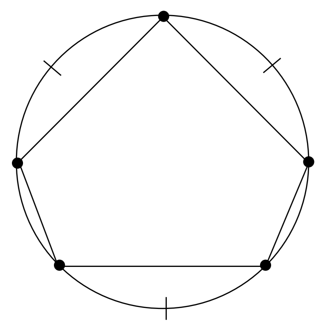


Figure 3.1: 10110110 in geometric notation. There are 8 Counts and 5 Notes.

Figure 3.2: Example of a long Palindrome in Haydn's Music

1. Find palindromes in binary notation, like 101101101 and compare them with Toussaint's palindromes. Would Toussaint consider your binary palindromes as palindromes?

Let's call Toussaint's palindromes **geometric palindromes** and the ones in binary notation **musical palindromes**. We want to understand how different these two really are!

> **2.** Independent Investigation: To compare the two palindrome definitions we want to record the number of geometric palindromes and the number of musical palindromes in two triangles, similar to Pascal's triangle. You can use Figure 3.4 and Figure 3.5. This will take some time, so work with another student on this problem and compare you results with other groups.

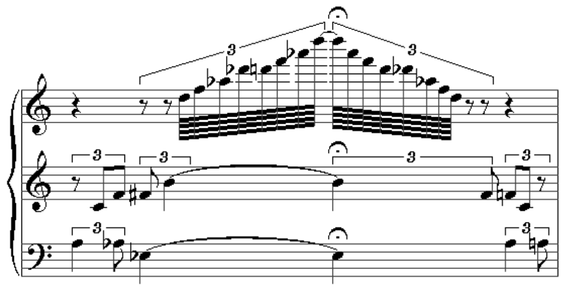


Figure 3.3: Example of a rhythmic Palindrome in Berg's Music

3. Do you notice any pattern in the triangles? Do you see any symmetry? Describe your observations.

4. Can you predict the entries of the next row of each triangle? Describe your strategies.

3.2 Patterns in the Palindrome Triangles

Did you notice that the addition pattern from Pascal's triangle works sometimes in the two new triangles? Try adding up two numbers and compare the result with the entry below. This seems surprising! Why does it sometimes work and sometimes not? To find out, we will start with the musical triangle.

5. Find a place in the musical triangle where the addition pattern works and write out the corresponding musical palindromes. Can you see how to create a palindrome in the lower row using the rows above? You can also fill in Figure 3.6.

6. Now try a place where the addition does not work. Write out the corresponding musical palindromes. Can you see what happens? You can also fill in Figure 3.7.

By now you have a good sense of how to count musical palindromes and why the addition patterns emerges from the triangle.

7. Independent Investigation: Look at examples for geometric palindromes to understand why the addition pattern sometimes works and sometimes fails. Explain your reasoning.

3.3 The Mystery of Palindromes

But the mystery of comparing the two kinds of palindromes is still unsolved...

8. Compare the two triangles and see if you can find a musical palindrome for a corresponding geometric palindrome. Which cases should be easy? Where do you get stuck?

Row Number = Number of Counts

Column Number = Number of Notes

Numbers start at zero.

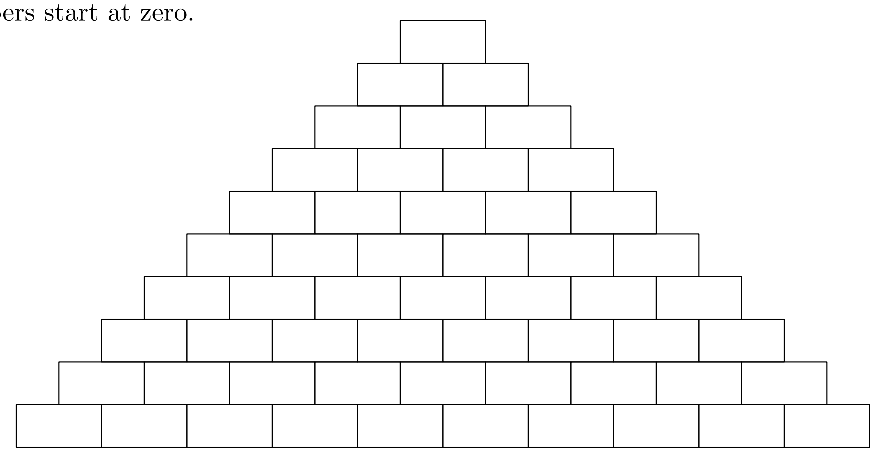


Figure 3.4: Musical Palindrome Triangle

Did you notice that all the entries in the triangles are the same except for the places where the musical triangle has zeros?

9. Why doesn't the geometric triangle have any zeros?

10. Why does the musical triangles have zeros?

Now the challenge seems to be that even if the numbers in the triangels agree, it is not clear how to take a musical palindrome and make a geometric one and vice versa! For instance, for 4 counts and 2 notes there are 2 musical palindromes and 2 geometric palindromes.

11. Can you see how to take 0110 and make it into a geometric palindrome? Be creative! You can use Figure 3.8 for your result. Write down your strategy.

12. Now make sure that your strategy works for all cases. Can you use your idea also to get a musical palindrome from a geometric palindrome? You can use Figure 3.9 to test your idea.

Row Number = Number of Counts

Column Number = Number of Notes

Numbers start at zero.

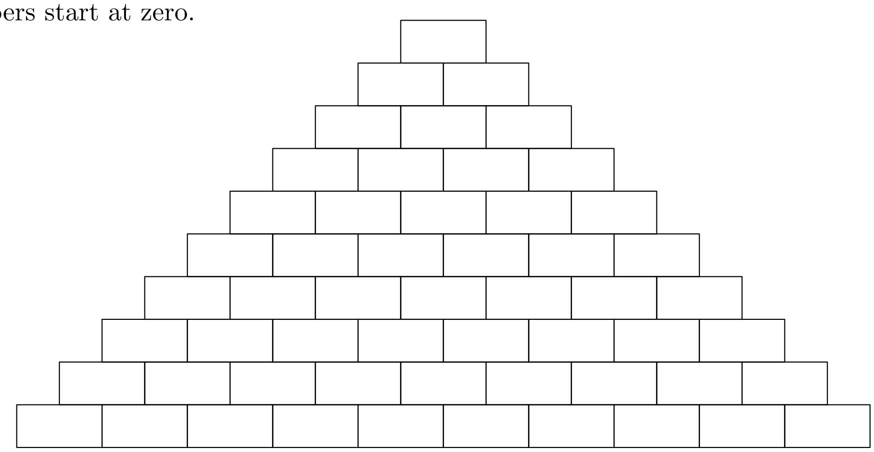


Figure 3.5: Geometric Palindrome Triangle

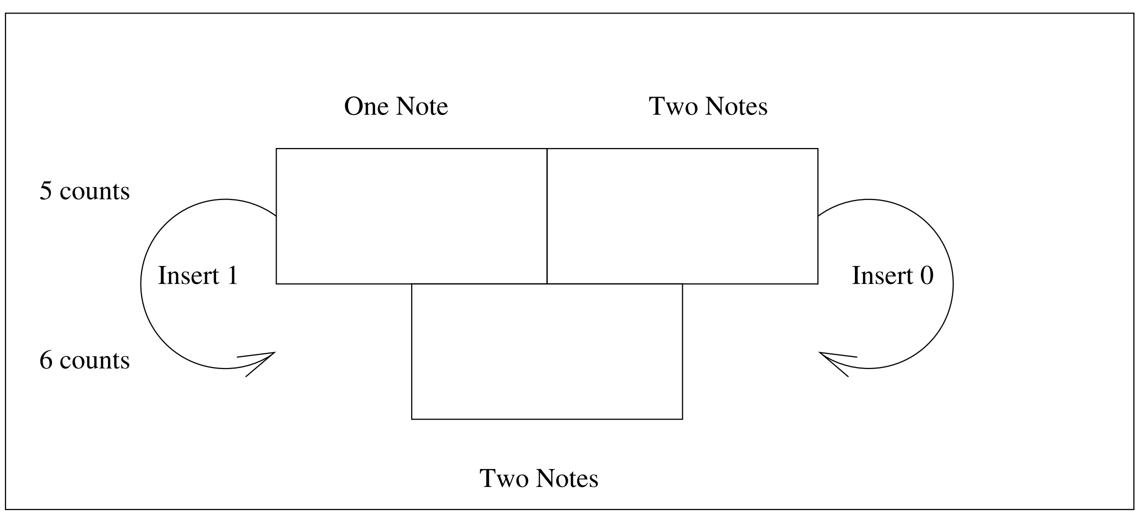


Figure 3.6: Addition in Part of the Musical Palindrome Triangle

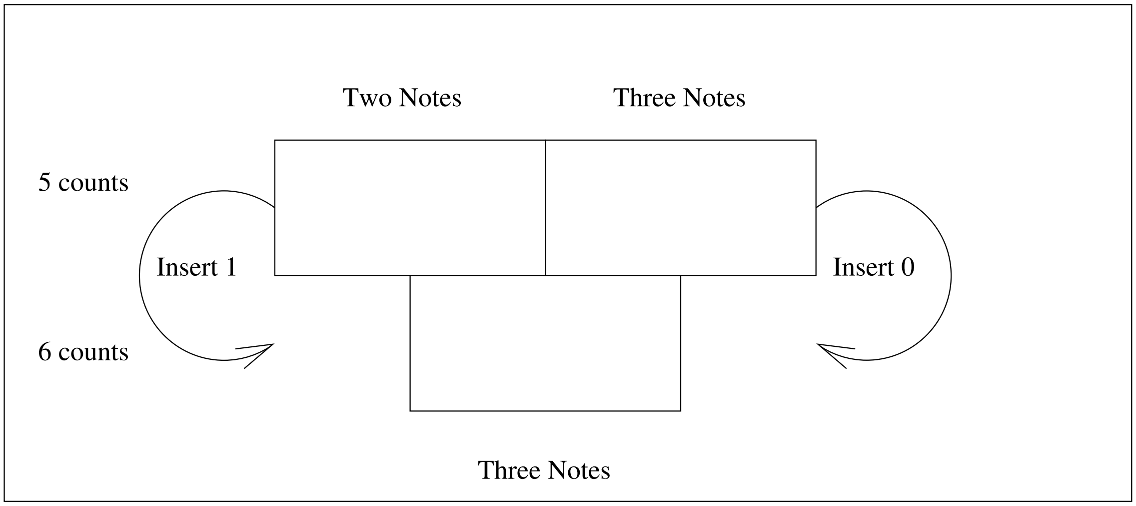


Figure 3.7: Addition in Part of the Musical Palindrome Triangle Fails

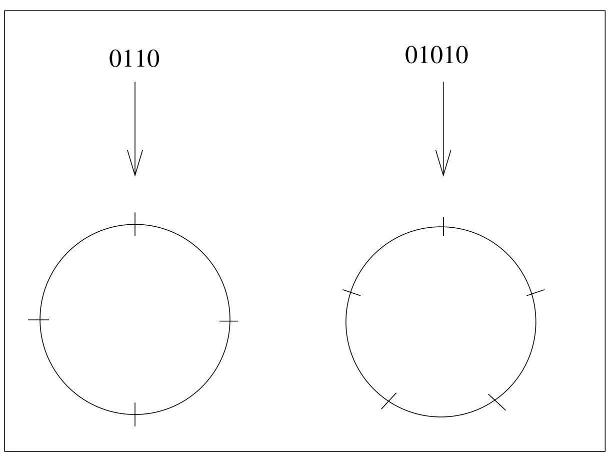


Figure 3.8: Musical to Geometric Palindrome

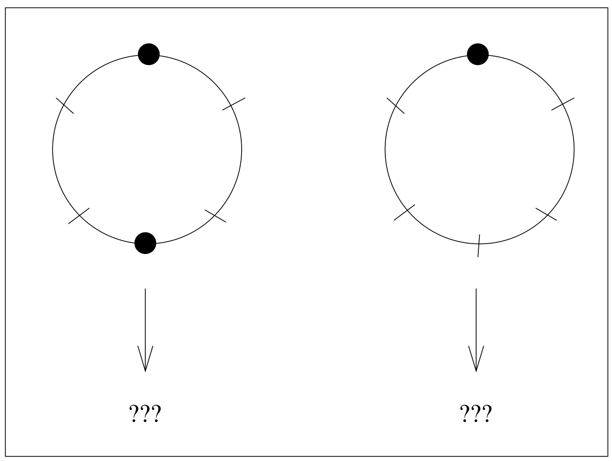


Figure 3.9: Geometric to Musical Palindrome

3.4 Further Investigations

3.4.1 Which palindromes are *better*?

We have thoroughly investigated two different kinds of rhythmical palindromes, but which one is *better*? And *better* in which sense?

13. Independent Investigation: Invent a piece of music, just using rhythms, that uses some palindromes and some rhythms that are not palindromes. You can use the software ABCdrums from Rachel Hall's Website `http://www.sju.edu/~rhall/Multi/drums.html` to help you play your piece. Now play your composition to several listeners and see if they can *hear* the palindromes. The purpose of this activity is to find out if we can hear palindromes at all, and if we can, which kind of palindromes are easier to hear.

3.4.2 Triangle Fractals

Look at Sierpinski's triangle in Figure 3.10. If you "zoom in" you can see that the triangle structure repeats itself over and over again. We call such ***self-similar*** objects ***fractals***.

14. Independent Investigation: Look at our musical and geometric palindrome triangles and see if you can find fractal-like structures. You might want to use color to emphasize special numbers or shapes.

3.5 Connections

Palindromes also play an important rule in other forms of art, for instance in poetry.

15. Find all palindromes in the poem "Dammit I'm mad" by **Demetri Martin** (American Comedian and Artist; 1973 -).

```
Dammit Im mad.
Evil is a deed as I live.
God, am I reviled? I rise, my bed on a sun, I melt.
To be not one man emanating is sad. I piss.
Alas, it is so late. Who stops to help?
Man, it is hot. Im in it. I tell.
I am not a devil. I level Mad Dog.
Ah, say burning is, as a deified gulp,
In my halo of a mired rum tin.
I erase many men. Oh, to be man, a sin.
```

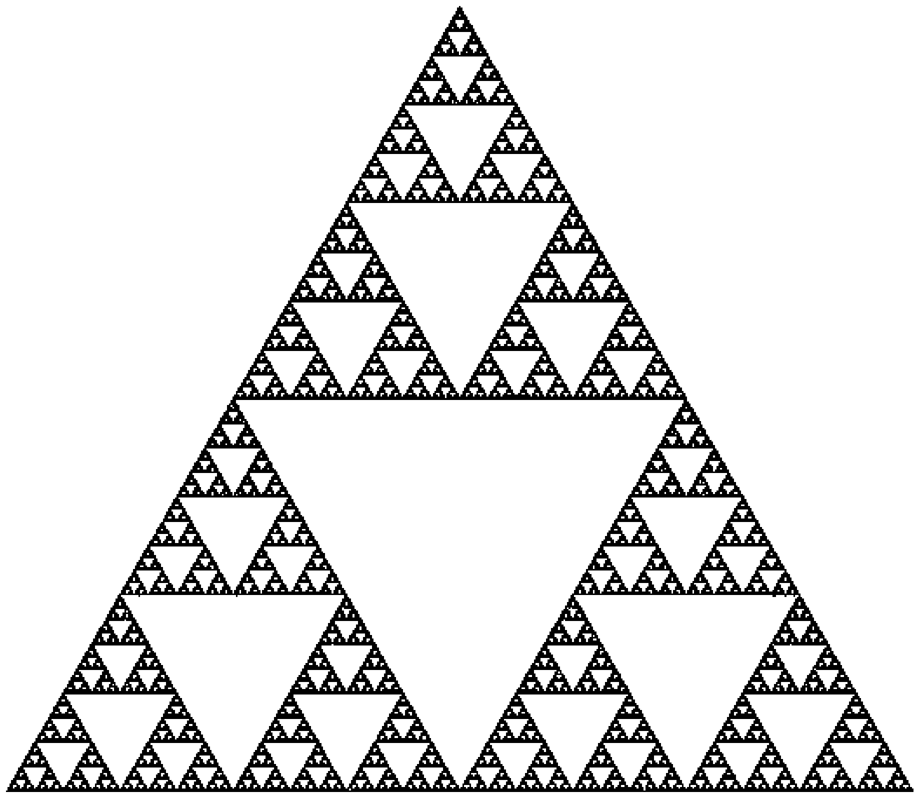


Figure 3.10: Sierpinski's Triangle.

```
Is evil in a clam? In a trap?
No. It is open. On it I was stuck.
Rats peed on hope. Elsewhere dips a web.
Be still if I fill its ebb.
Ew, a spider eh?
We sleep. Oh no!
Deep, stark cuts saw it in one position.
Part animal, can I live? Sin is a name.
Both, one my names are in it.
Murder? Im a fool.
A hymn I plug, deified as a sign in ruby ash,
A Goddam level I lived at.
On mail let it in. Im it.
Oh, sit in ample hot spots. Oh wet!
A loss it is alas (sip). Id assign it a name.
Name not one bottle minus an ode by me:
```

```
Sir, I deliver. Im a dog
Evil is a deed as I live.
Dammit Im mad.
```

"Weird Al" Yankovic's song *Bob* is a parody of "Subterranean Homesick Blues" by Bob Dylan and contains many palindrome phrases, see `http://www.youtube.com/watch?v=Nej4xJe4Tdg`.

In San Diego, you can walk across a bridge and play the 488 chimes while you walk. The song you hear was composed by Joseph Waters and plays the same in both directions to accomodate walking in either direction.

Figure 3.11: Bridge in San Diego with Palindromic Chimes.

Chapter 4

Tuning and Intervals

4.1 How perfect is Pythagorean Tuning?

> Sitting on the riverbank, Pan noticed the bed of reeds was swaying in the wind, making a mournful moaning sound, for the wind had broken the tops of some of the reeds. Pulling the reeds up, Pan cut them into pieces and bound them together to create a musical instrument, which he named "Syrinx", in memory of his lost love
>
> **Ovid** (Roman Poet; 43 BC - AD 18/19)

Have you ever watched someone tune a guitar? Or maybe even a piano? The lengths of the strings have to be adjusted by hand to exactly the right sound, by making the strings tighter or looser. But how does the tuner know which sound is the right one? This question has been asked throughout history and different cultures at different times have found different answers. Many cultures tune their instruments differently than we do. Listen for instance to the Indian instrument *sarod* in `http://www.youtube.com/watch?v=hobK_8bIDvk`. Also, 2000 years ago, the Greek were using different tuning ideas than we do today. Of course the Greek did not have guitars or pianos at that time, but they were still thinking about tuning for the instruments they had and about the structure of music in general. The **pan flute**, one of the oldest musical instruments in the world, was used by the ancient Greeks and is still being played today. It consists of several pipes of bamboo of increasing lengths. The name is a reference to the Greek god Pan who is shown playing the flute in Figure 4.1.

For the following investigations you need to make your own "pan flute" out of straws. Straws for *bubble tea*[1], work better than regular straws since they have a wider diameter. You need to plug the bottom with a finger to get a clear pitch. Put your lower lip against the opening of the straw and blow across the opening (but not into it). It helps to have some tension in the lips, as if you were making the sounds "p". Also, for shorter straws you need more air pressure than for longer straws.[2]

1. Take a straw and cover the bottom hole while blowing over the top hole. Practice until you can hear a clear note. *Why* do you think we hear a sound?

[1] "Bubble tea" is the American name for pearl milk tea from Taiwan. You need straws with a larger diameter to drink bubble tea, since the tea contains small balls made of starch.

[2] Tubes with diameter $\frac{1}{10}$ of their length are easiest to play!

Figure 4.1: Pan playing the pan flute.

2. Do you think the sound will be different if the straw is longer or shorter? Explain your thinking.

3. Take a rubber band, hold it tight between two hands and have someone pluck it. Can you hear a clear note?

4. Take a rubber band, stretch it over a container and pluck it. Can you hear a clear note? Why do we hear a sound?

5. Do you think the sound will be different if the rubber band is longer or shorter? Tighter or looser? Explain your thinking.

6. **Classroom Discussion:** How is sound generated? What exactly is vibrating? What is a *sound wave*? How do different musical instruments like drum, guitar, violin and trumpet generate sound?

For the next investigations we will use the modern piano as a reference tool, so that we can compare our sounds and give them labels. Even with the piano it is quite difficult to hear if two sounds are the same or not. If you have difficulties, turn to someone who has practiced music for a long time for support.

7. Take one straw and cut it such that it has the sound of any white key on a piano (except for the F and the B key, see Figure 4.2. We will discover later why these keys don't work here.) You can go to
`http://www.play-piano.org/play_online_piano_piano.html` to use the online piano.

8. Take a second straw and cut it so that it has a length of $\frac{1}{2}$ of the first straw.

9. Take a third straw and cut it so that it has a length of $\frac{2}{3}$ of the first straw. Be precise!

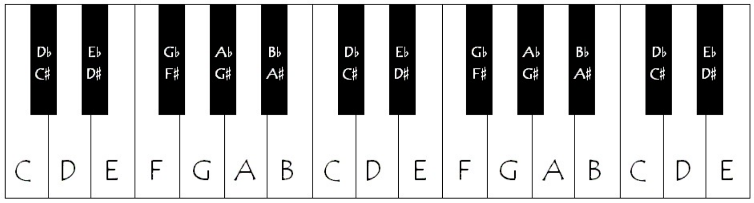


Figure 4.2: piano keys with labels.

10. Take a fourth straw and cut it so that it has a length of $\frac{3}{4}$ of the first straw. Be precise!

11. Compare the sounds of 2 straws at a time. We call two notes sounding at the same time an **Interval**. We write e.g. $(1, \frac{2}{3})$ for the interval of the first straw and the straw with length $\frac{2}{3}$. Listen carefully: which two straws sound the most alike? You can also sing the notes of the 2 straws and listen to the interval to make your decision.

12. **Classroom Discussion:** Share your intervals with the class. Decide together which fraction gives the "most alike" interval.

We call the interval that sounds the most alike an ***Octave***. Human brains seem to be hard-wired to perceive these sounds as alike or even the same. The thalamus is a part in the brain of mammals that is built in layers of neurons that correspond to octaves. See Figure 4.3. Additionally research shows that rhesus monkeys have "same" responses to melodies that are one or two octaves apart but "different" responses to other melody shifts.

Figure 4.3: Thalamus in the Human Brain.

This explains why we can find octaves in cultures all over the world even though their music may sound very different. Even though all cultures share octaves, there are many ways to divide the octave into smaller intervals. We call those choices ***scales***. In modern western culture, the major and minor scale are the most prominent scales. For example the C major scale corresponds to the white keys on a piano. Notice that on a piano you have to go up or down 8 white keys to travel an octave (starting on a white key and counting this first key as one of the 8).

You can go to `http://www.play-piano.org/play_online_piano_piano.html` to play the C-major scale. Take the intervals $(1, \frac{2}{3})$ and $(1, \frac{3}{4})$ and see if you can find the corresponding intervals on a piano.

13. Take your pair of straws for the interval $(1, \frac{1}{2})$. How many white keys are between the notes if you count the beginning and the end note as well?

14. Take your pair of straws for the interval $(1, \frac{2}{3})$. How many white keys are between the two straw-sounds if you count the beginning and the end key as well?

15. Why do we call the interval $(1, \frac{2}{3})$ a **fifth**[3]? Explain!

16. Why do we call the interval $(1, \frac{3}{4})$ a **fourth**? Explain!

You have probably heard of the mathematician and philosopher **Pythagoras of Samos** (Greek Philosopher and Mathematician; 570 BC - 495 BC), but did you know about the secret society called the ***Pythagoreans***? The Pythagoreans believed that *everything* in the world could be explained using mathematics, including music. There is not much evidence about the life of Pythagoras and his disciples, see Further Investigation 5. However, they are credited with some important discoveries in mathematics. The Pythagoreans believed that all music could be explained using mathematics. They used, for instance, the musical fifths to get to all other notes in their scales as the next Investigations illustrate. The tuning they used is called ***Pythagorean Tuning***.

17. Take the interval $(1, \frac{2}{3})$. Now take a third straw and cut it such that the length is $\frac{2}{3}$ of the previous $\frac{2}{3}$ straw. How much of your longest straw is your new, very short straw? Write your answer as a fraction and explain your reasoning.

18. What is the label of your new straw on the piano? Is it in the same octave as the first two straws? Can you see how to use the fraction to determine whether your new note is in the first octave or not? From now on we will call this octave (between our first two straws) our ***main octave***.

19. Compare the two fractions $\frac{1}{1}$ and $\frac{1}{2}$, whose sounds lie an octave apart. Which fraction operation do we have to do to get from one to the other? Explain how to go up and down octaves using fractions.

20. By looking at *any* fraction, how can you tell whether the corresponding note will be in the main octave or not? Explain your reasoning.

21. Take the fraction from Investigation 17. How can we use it to get a new fraction corresponding to the same note in the *main* octave?

22. You just found a fraction representation of a note in your main octave that corresponds to a fifth above a fifth. Continue the pattern by taking the next fifth and so forth. If you can't hear the sound of your straw anymore, see if you can find the mathematical pattern to continue this quest in theory. You should find a list of 5 fractions.

[3]We have to distinguish between the musical fifth (which is a specific interval between two notes), and a mathematical fifth (which is the fraction $\frac{1}{5}$.)

Figure 4.4: Medieval Woodcut showing Pythagoras.

23. Draw a number line from $\frac{1}{2}$ to 1 and label the first 5 fractions you found.

24. Look at a piano keyboard. How many steps are there in a fifth if you include the black keys?

25. We said earlier that a fifth corresponds to five white keys on the piano keyboard if you don't start from a B. Use Investigation 24 to argue why we had to exclude the B.

26. Using investigation 24, how many fifths do we have to go up on a piano keyboard before we return to the same note (some octaves higher)?

27. Now we will use the fraction $\frac{2}{3}$ to go up by fifths. Find the fraction representation of the note in the main octave that corresponds to 12 fifths above your original note. Explain your strategies.

28. How far is the fraction from investigation 27 from 1? Did you expect this answer? Explain.

29. Does the chain of fifths ever end? Use fractions to explain your answer.

30. Use the chain of fifths to explain problems that arise with Pythagorean tuning.

31. **Classroom Discussion:** Does the chain of fifths end or not? Compare your result of the fraction computation with the result on the piano keyboard. How perfect is Pythagorean tuning?

There are more problems with the Pythagorean tuning than the one you just discovered. A major *third*[4], e.g. the interval CE doesn't sound very nice in Pythagorean tuning. When thirds were used more frequently during the Renaissance and Baroque period, people abandoned Pythagorean tuning in favor of other tuning methods.

4.2 Frequency, Fractions and Ratios

The "height" of a sound is called ***pitch*** and our perception of pitch arises from the frequency of the sound. The frequency measures how fast the sound wave vibrates. In a long straw (big number) the air vibrates more slowly (small number) and in a short straw (small number) the air vibrates faster (big number), which means the length of the straws is inversely-proportional to the speed of vibration. For simplicity we will assume that the fractions for frequency are just the reciprocals of the fractions for length, i.e.

$$\text{frequency} = \frac{1}{\text{length}}.$$

For example a straw of length $\frac{1}{2}$ sounds with a frequency of $\frac{2}{1}$.

The unit of frequency is hertz (Hz), named after **Heinrich Hertz** (German Physicist; 1857 - 1894). 1 Hz means that an event repeats once per second.

Figure 4.5: Heinrich Hertz.

We want to redo the above investigations thinking about frequency instead of length.

[4] A major third consists of 2 whole steps, see page 36 for a definition of whole steps and Chapter 6 for more investigations about musical intervals and chords.

32. Write the intervals $(1, \frac{1}{2})$, $(1, \frac{2}{3})$, and $(1, \frac{3}{4})$ using frequencies instead of length.

33. By comparing the two frequencies that make our main octave, which fraction operation do we use to go up and down octaves? Explain.

34. Compute the ascending fifths as above using frequencies instead of length. Explain your strategies.

35. Draw a number line from 1 to 2. Label your first 5 frequency fractions.

36. Since the process of taking more and more fifths results in notes that sound out of tune, the Pythagoreans used the fraction $\frac{3}{4}$ to help them. Recall the key on the piano corresponding to the fourth, i.e. to the fraction $\frac{3}{4}$. How many fifths do we use to go up on the keyboard in order to get to the same note as the fourth (ignoring octaves)?

37. Why is it more accurate to work with the fourth instead of the fifths in investigation 36?

38. Label the frequency that corresponds to the fraction $\frac{3}{4}$ on your number line.

Your main straw could have been any length in the above investigations and hence correspond to any note from a white key (excluding B, of course). For the next section we will assume that it corresponds to the note C. The mathematics works out the same if you use another note as your starting point, but it makes it easier to read if we agree on a base note.

We want to discover how the Pythagorean fifths will give us the entire C-major scale!

39. Fill in the first row in table 4.1. If your main straw would correspond to the note C, how do the other frequency fractions we found relate to the keys on the piano? You can use the fractions you computed in the above investigations. Just match them with the C-major scale instead of the scale from your straws.

Table 4.1: Frequency Table

Note	C	D	E	F	G	A	B	C
Frequency Fraction	$\frac{1}{1}$							$\frac{2}{1}$
Ratios between Frequency Fractions								

40. **Classroom Discussion:** Compare the first row in table 4.1. Now look at the ratios[5] between adjacent fractions on your number line. Fill in row 2 in table 4.1. What patterns do you notice?

You just discovered the so called ***Pythagorean Tuning*** based on C. Unfortunately there are some problems with this tuning method... you will discover some of these in the next Investigations:

[5] To find the ratio between two fractions you need to divide one fraction by the other - you compute a fraction of fractions. We will divide the larger fraction by the smaller to make it easier to compare.

41. We tried to avoid the "incorrect" last fifth, also called the ***wolf interval***, by chosing the frequency $\frac{4}{3}$ instead of the last power of $\frac{3}{2}$. Will this solve the problem or will there still be a wolf interval? Explain.

42. Your piano is tuned in Pythagorean tuning based on C. Imagine you have a melody starting with the fifth CG. Do you think the song would sound "bad" if you started playing it on a different note? Explain.

So it seems that for some melodies the piano will sound in tune while for other melodies or other starting points of you melody it might sound out of tune. Musician would say: "If I played a song that uses a different ***key*** it would sound out of tune!". This *key* is not the same as a key on a keyboard. It is an abstract term roughly describing a set of notes that a piece of music is most likely to use. You can for instance say that a song is being played in the key of "C major".

That is not what we wanted! It gets even weirder:

43. Musicians call the step from one key to its closest neighbors (can be black or white) a ***half step***. Two connected half steps are called a ***whole step***. For example, CD is a whole step, while $CC\sharp$ is a half step, as is EF. Compare the ratios for a half step and a whole step in Pythagorean tuning (table 4.1). What do you notice? Are two half steps really a whole step? Remember to use ratios and differences in your argument.

44. Why is Pythagorean tuning a very natural way of tuning, even though problems arise?

4.3 The Roots of Equal Temperament

Since the Pythagorean tuning is not the same for all *keys*, other ways of tuning were developed over time. In the 18th century ***well tempering*** was used, in which compromises were made such that every *key* would sound good but slightly different. One advantage of each *key* sounding different is that the mood of a piece of music can be expressed by the choice of *key*.

Since the middle of the 19th century ***equal temperament*** is most commonly used. This tuning requires a new mathematical idea which you will discover in the next Investigations. We know that the frequency interval $(1, 2)$ gives us an octave. It is customary in Western Music to have 12 steps in an octave. Therefore we need to find a way to split the interval between 1 and 2 into 12 "equal" steps. Since we are dealing with ratios here, we need all the steps to have the same ratio. Look back at table 4.1 to see 7 steps (ratios of frequency fractions) that are not all equal.

45. Split the interval between 1 and 2 into 2 "equal" steps such that the *ratios* are the same. This means we are looking for a fraction, say x, between 1 and 2, such that the ratio of 2 and x is the same as the ratio of x and 1. What is x? Describe your strategy.

46. Compare your solution with the following problem: Split the interval between 1 and 2 such that *differences* are the same. This means we have to find a number, say y, between 1 and 2 such that the difference between y and 2 is the same as the difference between y and 1. What is y? Did you get the same answer as in the last investigation?

47. Classroom Discussion: Compare the two solutions above to get "equal size" steps in the interval $[1, 2]$. Compare your strategies. What does "equal size" mean? Compare your results. Now go back to Investigation **23** and Investigation **35** and explain why we did not see any useful spacing pattern on the number lines.

48. Split the interval between 1 and 2 into 3 steps with equal ratios. Describe your strategy.

49. Split the interval between 1 and 2 into 4 steps with equal ratios. Describe your strategy.

50. Split the interval between 1 and 2 into 5 steps with equal ratios. Describe your strategy.

51. Split the interval between 1 and 2 into 12 steps with equal ratios. Describe your strategy.

52. Summarize how to find the frequencies for the *equal temperament tuning.*

53. What are some advantages and some disadvantages of *equal temperament tuning*?

You really understand Pythagorean tuning and equal temperament tuning now, and you have traveled through many centuries of music and mathematics history. Hidden in the above mathematics is some history about numbers:

The Pythagoreans believed that *every* number could be written as a fraction. Mathematicians call these numbers **Rational Numbers**. According to legend **Hippasus of Metapontum** (Greek Philosopher; 500 BC -) was put to death by Pythagoras because he had revealed the secret of the existence of ***irrational numbers***: numbers that can not be written as fractions.

Figure 4.6: Hippasus of Metapontum.

It might seem easy to grasp for us now, but every time mathematicians expand their ideas of numbers it is like a small revolution. And there are more than just irrational numbers! There are for instance *complex numbers* and *imaginary numbers* and *surreal numbers.* For the latter you can read the book Discovering the Art of Mathematics: The Infinite.

54. Do you find it surprising that the Hippasus was put to death?

55. Name one irrational number. Do you know more?

4.4 Further Investigations

There are other ways of tuning that you have not discovered yet.

F1. Research the mathematics behind meantone tuning. How is it different from Pythagorean and equal temperament tuning? How is it similar?

F2. When (and why) was meantone tuning used?

The way Greek mathematicians first encountered irrational numbers was not in music, but in geometry. You will solve their problem in the next Investigations.

F3. In a square with side length equal to 1, what is the length of the diagonal?

F4. Find a proof of the fact that $\sqrt{2}$ is an irrational number. You can look at books or go online. Explain the proof to someone else without looking at your notes to see if you fully understand it.

4.5 Connections

F5. Read "The Ashtray: Hippasus of Metapontum (Part 3)" by Errol Morris published in the New York Times Opinionator. What do we *actually* know about Hippasus?

F6. Figure 4.7 shows graphs of waves with different frequencies. How does this relate to waves of air in the straws?

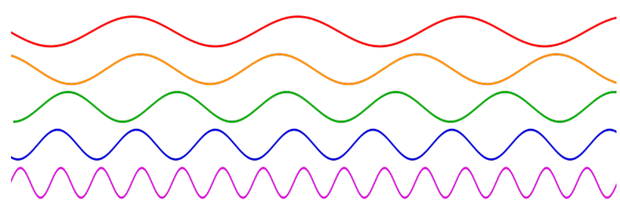


Figure 4.7: Graphs of waves with different frequencies.

F7. Check out Ruben's Tube videos on `youtube.com`. How does this connect to graphs of sound waves? See Figure 4.8.

The following video link shows a fascinating 2-dimensional "Ruben's plane" with flames lighting up to music:
`www.iflscience.com/physics/amazing-2d-rubens%E2%80%99-tube-visualizes-sound-plane-fire`

F8. Different cultures at different times also used varying scales for their music. In Timothy Johnson's book [9], you can investigate (diatonic) transposing patterns for different scales. Proving why these patterns occur is challenging and really fun.

Figure 4.8: A Ruben's Tube Experiment.

Chapter 5

Fractal Music

> The most complex object in mathematics, the Mandelbrot Set ... is so complex as to be uncontrollable by mankind and describable as "chaos".
>
> **Benoit Mandelbrot** (French American Mathematician; 1924 - 2010)

5.1 Fractals

Among the most beautiful images mathematics can create are images of fractals, see Figure 5.1. Go to `http://www.youtube.com/watch?v=G_GBwuYuOOs` and watch the Mandelbrot set zoom in closer and closer.

1. Looking at the Mandelbrot fractal, do you think mathematics can be beautiful?

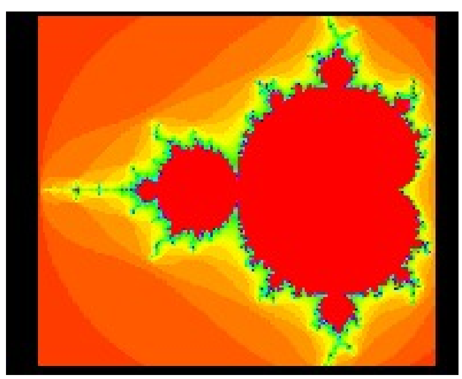


Figure 5.1: Mandelbrot Fractal

First, we want to see what the secret of fractals is. Construct the following three examples of fractals. Comparing them, what do you think is special about fractals?

2. Skyline:

Construct a line segment. Divide it into proportions x:y:z (going from left to right) where x+y+z=1. You can make a simple choice like $x = y = z = \frac{1}{3}$ or a more interesting one like $x = 0.3,\ y = 0.5,\ z = 0.2$. Construct a square on the middle portion. Now divide each of the three horizontal line segments into the same three proportions and construct squares on the middle portions. Continue until you see a nice "skyline".

3. Koch's Snowflake:

Construct an equilateral triangle (all sides have the same length). For one of the ***edges*** (mathematicians call the sides of a triangle edges) of the triangle, divide the edge into three equal parts. Construct an equilateral triangle on the middle third of the line and then erase the base of that triangle. Repeat this process on each of the four line segments. Repeat this process for all edges of the original triangle. You will see a nice "snowflake".

4. Sierpinski's Triangle:

Draw an equilateral triangle. Connect the midpoints of the edges. Ignore the new middle triangle that you get and repeat the process for the other three triangles. Repeat this process until you see a beautiful pattern of triangles emerge.

5. Explain what a fractal is given the examples you have seen so far.

You have seen that a fractal is an object that displays ***self-similarity***. This can be true for an object from nature, like cauliflower or fern, or for a more abstract mathematical objects like Koch's Snowflake see Figure 5.2.

We want to create a fractal that has different replacement rules. Let's make a different skyline where you start with $x : y : z$ such that $x+y+z = 1$ but now we draw an arc on top of the middle piece and a skyline piece on the first and last section. This is our starting point, see Figure 5.3. Now, whenever you see a straight line you replace it with the skyline as before (erecting a square on the middle section), but whenever you see a arc, you place another semicircle on top of it with some distance you can decide on. Do you like this skyline?

6. Come up with your own skyline! Draw a picture and explain your replacement rules.

5.2 Musical Fractals via L-Systems

Now we have played with different fractal structures but the big question is: Can we somehow take this idea of self-similarity and translate it into music?

> **7.** INDEPENDENT INVESTIGATION: Compose your own piece of fractal music! Be creative. Explain in detail how you used the idea of a fractal.

Building on the idea of C. Hazard and C. Kimport, see [8], we want to to create music using fractals via L-Systems: An **L-System** is a method of generating long strings of symbols from a

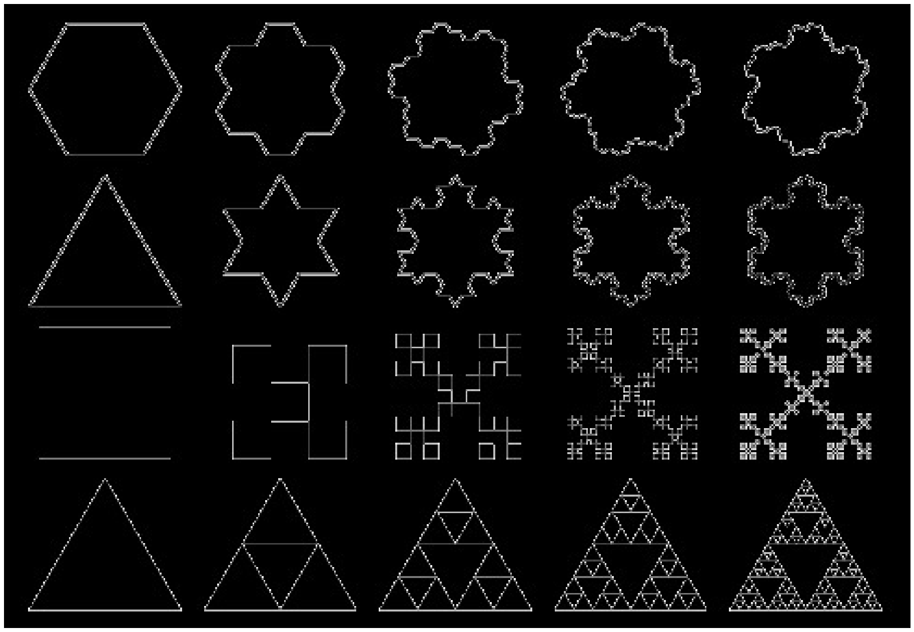


Figure 5.2: Abstract Fractals like Koch's Snowflake and Sierpinski's Triangle

short initial string (or axiom) and a set of production rules, one for each symbol. From here, we generate longer strings by replacing each symbol with its respective rule, and repeat this process until we have a string of a desired length. Here is an example:
Axiom:

$$AB$$

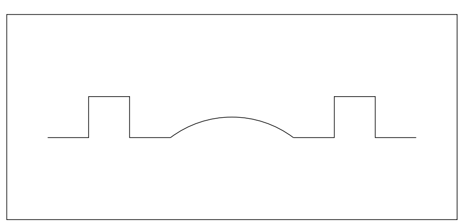


Figure 5.3: New Skyline starting point.

Production Rules:

$$\begin{aligned} A &\rightarrow ABC \\ B &\rightarrow CAD \\ C &\rightarrow DC \\ D &\rightarrow BDB \end{aligned}$$

Start out with the axiom, "A B". The production rules tell us to replace A with ABC and B with CAD. Therefore the string would become "A B C C A D".

8. If we use the same production rules with this new string, what do we get? What do you get if you do it one more time?

9. Independent Investigation: We want to translate these strings into music. Can we read the letters as names of notes? Or names of chords? How about names for lengths of notes? Choose your favorite method and create your own piece of music. You can use the software *finale notepad* `http://www.finalemusic.com/notepad/` if you are familiar with the standard music notation to help you create and play your piece. If you are not familiar with reading and writing music you can compose a rhythm instead using ABCdrums from Rachel Hall's Website `http://www.sju.edu/~rhall/Multi/drums.html`.

10. Classroom Discussion: Play your piece of music to your group or the class and share your fractal composing method with them. What is similar and what is different about the compositions in your group?

5.3 Musical Fractals using Turtle Graphics

You have used a mathematical idea to create music but we are missing the connection with the images that were so appealing about fractals. So let's create a musical fractal next that is based on an image of a mathematical fractal.

In the Turtle Graphics interpretation, there are four basic symbols:

F = Move forward one unit and draw a line while you are moving
f = Move forward one unit but don't draw a line
$+$ = Turn d degrees to your left
$-$ = Turn d degrees to your right

As before we can choose an axiom and assign production rules using these 4 symbols, but now you have to also pick d, the number of degrees. Pick a point on your page where you will start drawing and a direction you are facing (so that you know which direction you will draw in next!).

11. Independent Investigation: Can you figure out which choice of axiom and production rules will give you Koch's snowflake? Here are some hints: Think first about which symbols give you an equilateral triangle. Do you know how big the angles are in a triangle with sides of equal length?

Pick d as this angle and remember that d has to stay the same throughout your axiom and production rules.

Now we want to find the production rules. It may be easier to first find the rule that creates one edge, as shown in Figure 5.4. Try out if your turtle graphics creates the correct fractal!

Can you now find the axiom and rule for Koch's Snowflake[1] ?

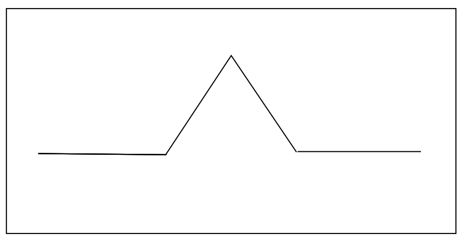


Figure 5.4: The first step to Koch's Snowflake...

Turtle graphics is an interesting name for the procedure described above. Do you have any idea why it is called turtle graphics? Look at Figure 5.5.

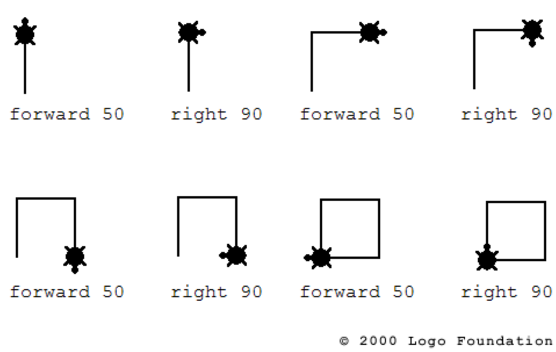


Figure 5.5: LOGO and Turtle Graphics

Where is the music connection? Instead of interpreting the symbols as notes or chords, they are interpreted as instructions to "draw" a melody on the staff. In our musical interpretation, horizontal motion is seen as note length, while vertical motion is seen as change in pitch. We can choose a starting note and a length that corresponds to the geometric unit. Also we can decide how we want to move "up" on the staff, by half notes or whole notes or by staying within a scale. So there is still lots of room to play![2]

12. Using the above method, create a melody that corresponds to Koch's snowflake. Play your melodies to your group and compare. If you are unfamiliar with playing music, use the computer to help you perform your piece.

5.4 Finding Fractals in Music

We found different ways to use the idea of fractals to create music. How about music that already exists? Can we find fractals in music? Of course not all musical pieces show fractal-like structures, but there are some famous ones that do.

13. Consider the following rhythm in Figure 5.6. Can you find a repeated use of the pattern *AAB* where each *B* section lasts twice as long as each *A* section? How many are there? Describe how this reminds you of a fractal.

Figure 5.6: Finding Fractal Structure in Sheet Music

Now we are ready for a famous example: Bach's Cello Suite 3, see Figure 5.7.

14. Harlan J. Brothers, [2], detected repeated use of the pattern *AAB* on different scales, where each *B* section lasts twice as long as each *A* section. See Figure 5.7. Can you find those sections? How does the structure of the piece resemble a fractal? Explain.

[2] If you only choose notes from one scale, for instance the pentatonic scale, your piece is more likely to sound traditionally pleasing than if you choose half-note steps.

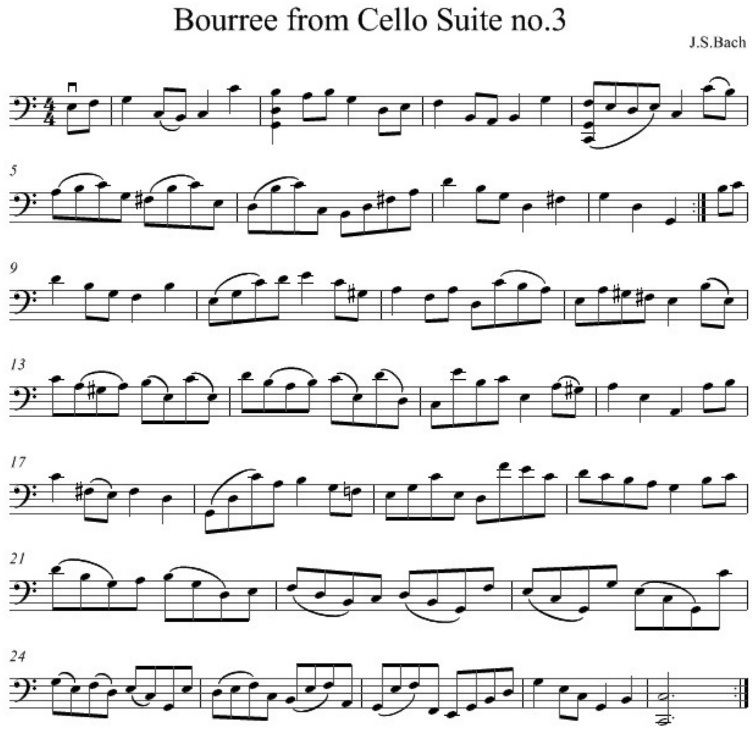


Figure 5.7: From Bach's Cello Suite

5.5 Connections

Investigate Discovering the Art of Mathematics: Geometry to play with dimensions of fractals.

Investigate Discovering the Art of Mathematics: Calculus to see if the area inside a fractal can be finite or not.

Chapter 6

The Space of 2-Chords

6.1 Maximally Even Chords

> Music is architecture translated or transposed from space into time; for in music, besides the deepest feeling, there reigns also a rigorous mathematical intelligence.
>
> **Georg Hegel** (German Philosopher; 1770 - 1831)

Do you know what a chord is? A **chord** is a number of notes sounding at the same time, there could be a 2-chord with 2 notes sounding but also a 5-chord with 5 notes sounding simultaneously. 3-chords are the most popular ones, you might have seen them written for guitar or piano players above the lyrics of a song, for instance C major or A minor.

The kinds of chords and scales that people found pleasing to listen to changed over the centuries. For instance minor and major ***thirds*** (2 notes that are 3 or 4 half steps apart, e.g. $CE\flat$ or CE) which are now part of basically every piece in Western Music where uncommon before the 15th century. Chords consisting of 4 notes became popular in the 17th century. It is interesting to notice how much music, art and mathematics change over time.

Now let's say you want to know how many (different) chords there are. How would you do that? Is there a connection to our questions in Chapter 2 about counting possible rhythms? You will notice that there are a lot of possible chords! What if I didn't want to count all of them but just very special chords?

First we need to talk a bit about music, since not all readers have a background in music. There are 12 notes in traditional western music:

$$C \quad C\sharp = D\flat \quad D \quad D\sharp = E\flat \quad E \quad F \quad F\sharp = G\flat \quad G \quad G\sharp = A\flat \quad A \quad A\sharp = B\flat \quad B. \tag{6.1}$$

We say **C sharp** for $C\sharp$ and **D flat** for $D\flat$. Those two notes sound the same[1] but musicians like to distinguish between them for historic and harmonic reasons. In this chapter we will ignore this distinction. Sometimes we will also use numbers instead of notes:

$$C = 0,\ C\sharp = 1, \ldots, B = 11.$$

We call the distance between two notes in our list (6.1) a **half step**.

[1] in equal temperament

After 12 half steps (an **octave**) the notation repeats, because we hear those notes as very similar. See Chapter 4 for more information about octaves and pitch. For the purpose of thinking about chords we won't distinguish between a higher sounding C and a lower sounding C. In our introduction, Chapter 1, you will find an explanation for the standard music notation.

If we identify all notes that represent the same sound up to octaves, we can think of the notes living around a circle, repeating over and over again, see Figure 6.1.

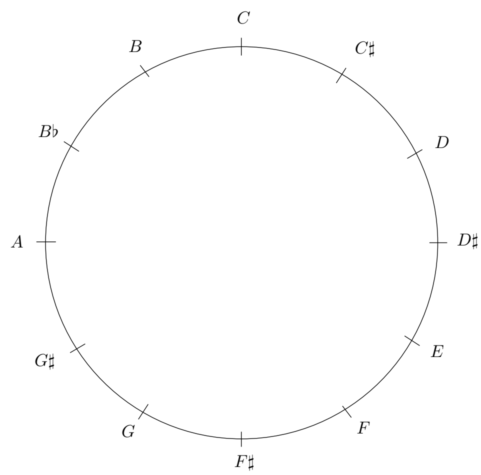


Figure 6.1: Writing the Notes around the Circle.

As a warm-up, we want to find all 3-chords that are "the most evenly spread out" around the circle. This property is called **maximally even** and was first described by **John Clough** (Music Theorist; -) and **Jack Douthett** (Mathematician; -) in 1991.

1. INDEPENDENT INVESTIGATION: How many different arrangements for evenly spread out 3-chords are there total? Use Figure 6.2 to draw your solution. How about 4-chords, 5-chords, ...? Do you notice any pattern?

If you want to investigate more about maximally even chords, try the exercises in T. Johnson's book, [9].

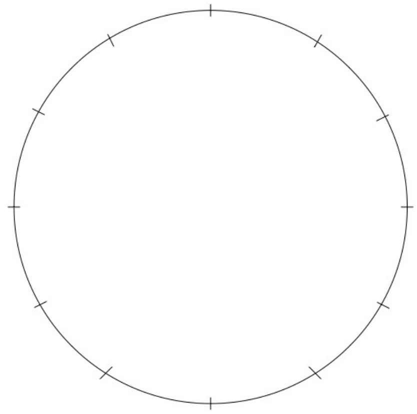


Figure 6.2: Draw your Table Arrangements

6.2 Chord Geometries and The Space of 2-Chords

> Mathematics and music, the most sharply contrasted fields of scientific activity which can be found, and yet related, supporting each other, as if to show forth the secret connection which ties together all the activities of our mind, and which leads us to surmise that the manifestations of the artist's genius are but the unconscious expressions of a mysteriously acting rationality.
>
> **Hermann von Helmholtz** (German Physician and Physicist; 1821 - 1894)

Figure 6.3: Dmitri Tymoczko

Dmitiri Tymoczko is a Music Professor at Princeton University, see Figure 6.3, who has discovered how to represent the universe of all possible musical chords in graphical form, [14]. In this section we will investigate his program and figure out the mathematics behind his wonderful ideas.

Please go to the following link online and download the free software ChordGeometries by Dmitri Tymoczko `http://music.princeton.edu/~dmitri/ChordGeometries.html`.

After you start the program ChordGeometries, go to *Geometries* and click on *Dyadic Space*. A new window called *mobius* will open showing you a space of 2-chords, see Figure 6.4. If you

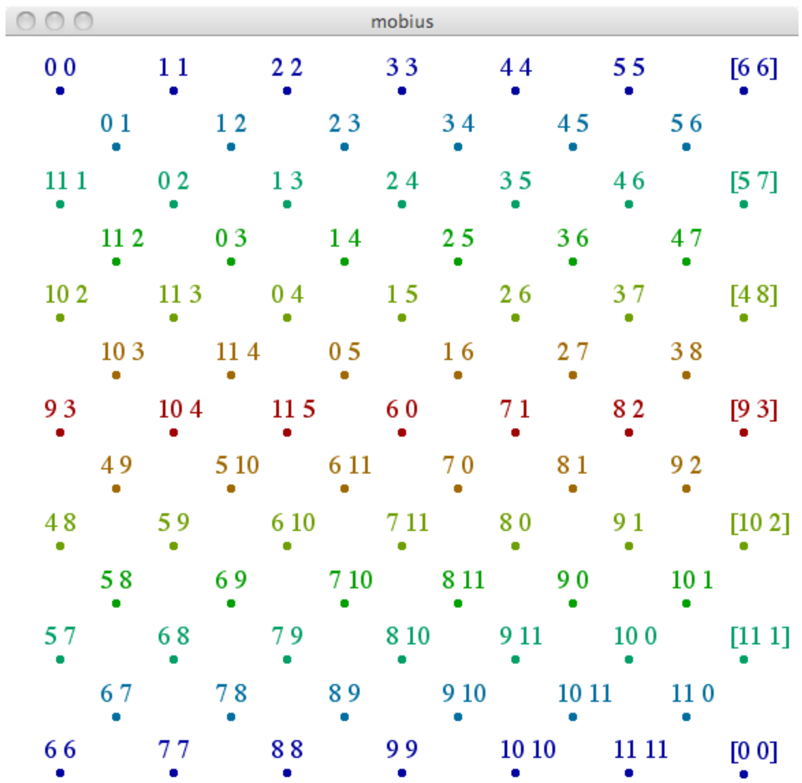


Figure 6.4: Dyadic Space

click on two keys of the piano, one of the points in the Moebius window will light up. Explore the space by choosing different 2-chords and watching the point move in the mobius window.
On the above webpage you will find a link to a movie clip called "Deep Purple", which plays the music while you watch the 2-chords move through space. There are a few questions that come to mind (we will answer them later in the investigations, for now just read them to see where we are going):

1. Every chord (point) has two numbers attached to it, how do we know which numbers correspond to which chord?

2. When you type in two chords after each other, say C E on the lower part of the piano and

an octave like $G\ G$ on the upper part, the path that the dot takes in the mobius window is rather complicated. What is happening?

3. The points in space have different colors, from red over orange, yellow and green to blue. What is the significance of the colors?

4. Why is the window called "mobius"?

Many mathematicians prefer to deal with numbers rather than the musical letters, and the convention is to identify $C = 0,\ C\sharp = 1, D = 2,\ D\sharp = 3,\ E = 4,\ F = 5,\ F\sharp = 6,\ G = 7,\ G\sharp = 8,\ A = 9,\ B\flat = 10,\ B = 11$. What happens with the next C? We should call it 12 but on the other hand we want every C to be the same when we think about chords. Mathematicians found a solution for this problem, they just say that 12 is ***equivalent*** to 0 and write $12 \equiv 0$.

2. If that is true, then what is 16 equivalent to?

3. If you would add two numbers what is the result equivalent to? For example, $8+4 = 12 \equiv 0$. Can you compute $9 + 6 \equiv$?

This new way of computing leads to an area in mathematics called ***Modular Arithmetic*** which is used heavily in many parts of mathematics, for instance in number theory and in algebraic geometry. You are actually using it every day, whenever you think about the clock! If you add 5 hours to 8 o'clock you get 13 o'clock but we usually call it 1 o'clock.

4. Did we just answer our first question? Go back to the program and try out if you understand which number pair a 2-chord represents.

Do you remember how to draw the graph of a function using an x-axis and a y-axis? We will do something similar here. Let's take the 2-chord $C\sharp\ E$, which corresponds to the number pair 1 4. We will draw two axes, the x-axis horizontally and the y-axis vertically. They meet at $x = 0,\ y = 0$ which is called the **origin**. Start at the origin and go 1 to the right and then 4 up and draw a point. This is your point in space which we label 1 4, see Figure 6.5.

5. Find the points for the 2-chords $G\ G$, $C\ G$ and $A\ D\sharp$ in the graph, Figure 6.5.

6. What happens if we continue on the x-axis towards the right, and we pass 11? What if we travel to the left of the origin? Take a piece of paper and label it with more numbers on the x-axis, at least 12 numbers to the left of the origin and 12 to the right of the origin. Now hold up the paper and press the points together that should be identified on the x-axis because they are equivalent. What shape do we get?

The next step is a bit tricky with paper, because we can't stretch it as we would like. So either you have to do this in your head or you can use a slinky to visualize the next step, see Figure 6.6.

7. Take your paper and label it additionally with more numbers on the y-axis, at least 12 numbers above the origin and 12 below the origin. Bring it back into the shape that identifies all the points on the x-axis and now, looking at the tube of paper in your hand, you have to identify the corresponding points on the y-axis. What shape do you get? Can you draw a picture?

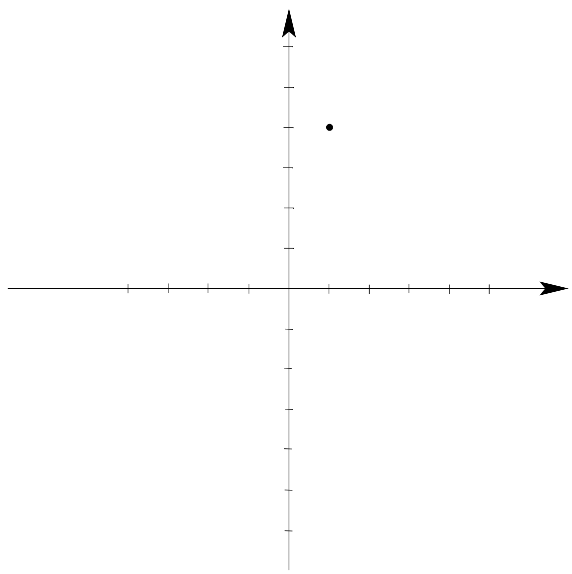


Figure 6.5: Graphing the point $(1, 4)$ in the xy-plane.

Mathematicians call this shape a ***torus***. From the whole infinite 2-dimensional plane we really only need the section that covers $0, \ldots, 11$ on the x-axis and $0, \ldots, 11$ on the y-axis, since everything else overlaps with this region. We call this the ***Fundamental Domain***, see Figure 6.7.

This is great as an image for the space of 2-chords but doesn't quite explain the mobius window from ChordGeometries. So what are we missing?

8. Compare the numbers on our fundamental domain and the number in the mobius window. What is the difference?

You probably found that we need to identify more points in our fundamental domain. We (the authors) decided to work with the triangle underneath the diagonale in our fundamental domain. See Figure 6.8. All the 2-chords in that triangle are the same as the ones above the diagonal, if we ignore the order in which the notes are played. For instance 1 9 is the same as 9 1. Unfortunately, when we look at just the lower triangle, we still don't have the same region as in the mobius window. What is happening? Here is a hint: Look at Figure 6.9. We cut our triangle under the diagonal into two pieces and compare the right one of the two pieces with the triangle that you can see under the x-axis. Convince yourself that the 2-chords in these two areas are the same (up to order)!

Figure 6.6: Slinky

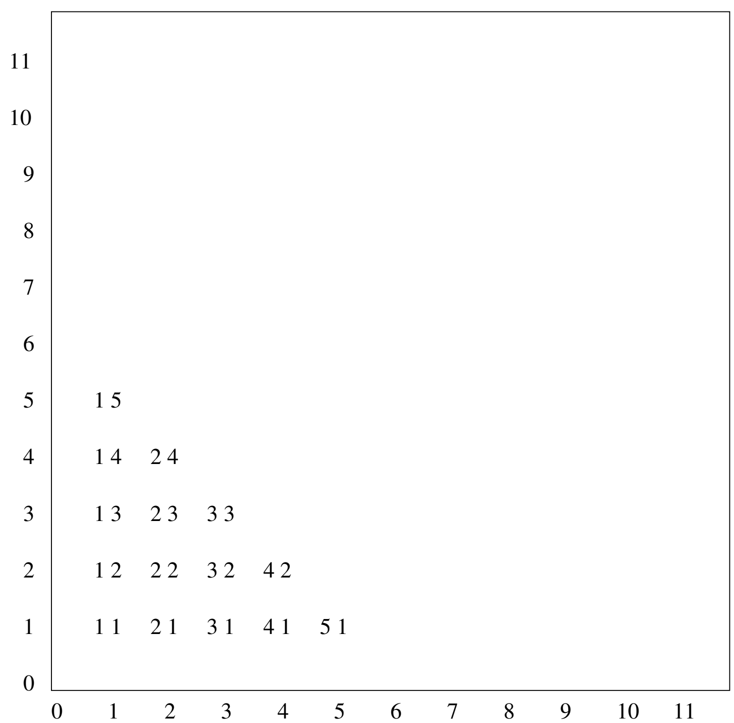


Figure 6.7: Fundamental Domain of Musical Torus

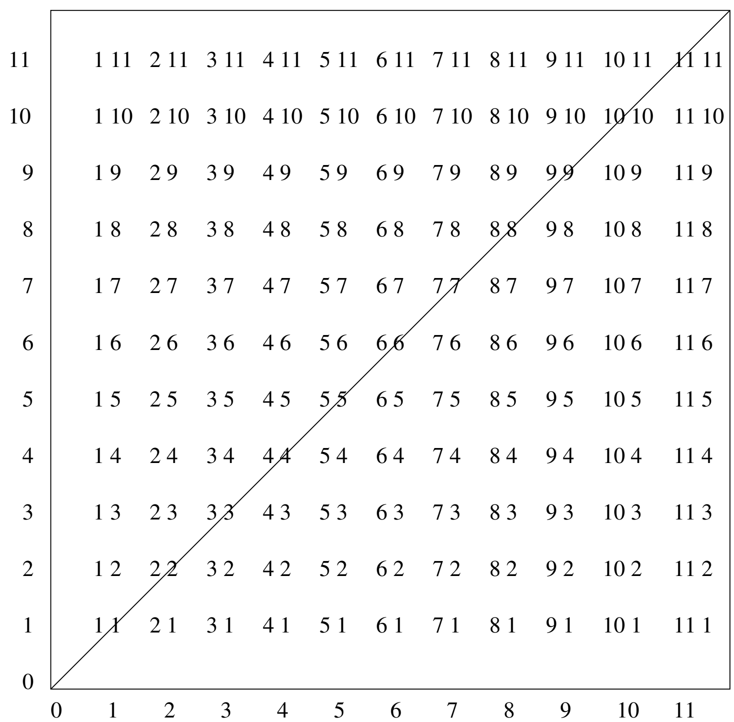


Figure 6.8: Fundamental Domain of Musical Torus with Triangles

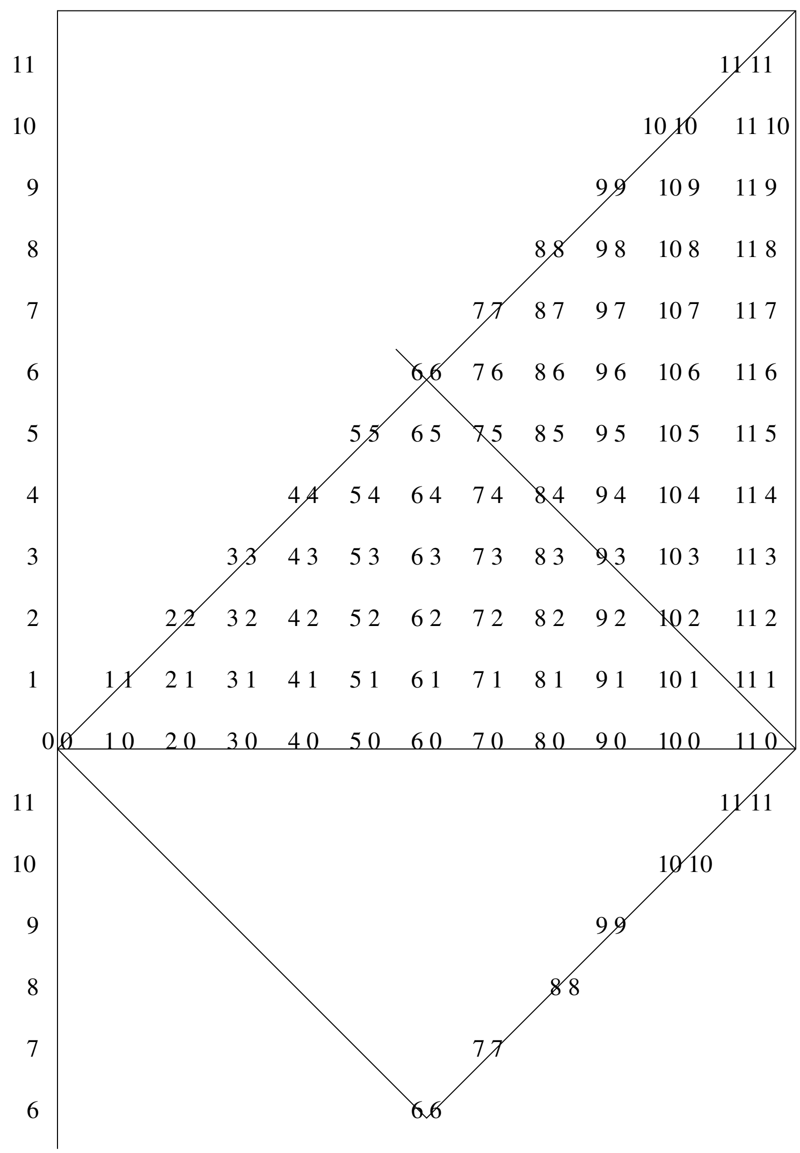


Figure 6.9: Fundamental Domain of Musical Torus with more Identifications

Can you see the mobius window now? All this cutting and moving might make you wonder...

9. Cut out the square that has exactly the number pairs from the mobius window. Can you see how two of its edges (sides) have the same number pairs? Glue the paper to itself connecting all equal number pairs. Describe the shape you get.

The loop that you get was invented by **August Ferdinand Moebius** (German Mathematician and Astronomer; 1790 - 1868) in 1858, although it was independently discovered by **Johann Benedict Listing** (German Mathematician; 1808 - 1882), who published it, while Moebius did not. It is famous for its property of having only one side.

10. Take a pen and draw a line starting anywhere on the strip, continuing along the strip. Will you connect again with your original line? Why or why not?

Mathematicians say that the Moebius strip is ***non-orientable***. See Figure 6.10. As mathemati-

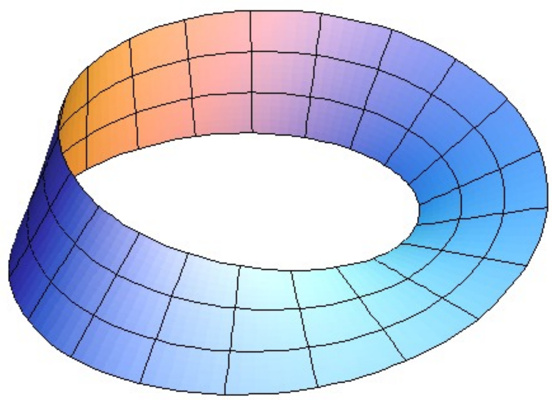


Figure 6.10: Moebius Strip

cians we are excited to realize that the space of 2-chords is the same as the famous Moebius strip!

We answered two of our original four questions, but the others should be easier, now that we understand the space of 2-chords better.

11. In question 2 on page 52 we were wondering why the path between some points in space is so complicated. Can you answer this question now? It might help to look at the Moebius strip instead of the Moebius window.

12. The last puzzle is the choice of colors in the moebius window. Here are two hints:

- Download the first Chopin video clip from the ChordGeomtries download page and notice the choice of colors.
- Go back to Section 6.1 and connect your results from there with our color problem here.

We answered our 4 questions about the ChordGeometry program and have now a much deeper understanding of the geometrical view on musical chords.

6.3 Further Investigations

F1. How could a musician benefit from the knowledge of the geometry of chords? Why is this research interesting to a mathematician? If you don't know an answer interview some professors!

F2. We want to play some music on the torus! If you take a curve (i.e. a curved line) on the torus, you can think of it as a possibly curved line in your xy-coordinate system. Every point corresponds to a 2-chord, so really your curve on the torus is a sequence of 2-chords.

Draw the two curves in Figure 6.11 in the xy-plane and then play the corresponding 2-chords on a piano.

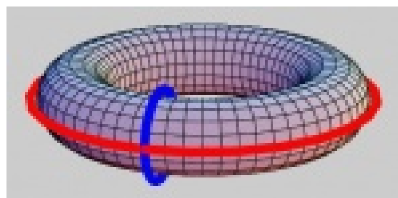


Figure 6.11: 2 Curves on a Torus

F3. Now draw the diagonal $x = y$ in the xy-plane. What does the curve look like on the torus? Play the 2-chords on the piano.

F4. Compose your own torus music by creating a curve on the torus and playing the 2-chords. Enjoy.

Chapter 7

The Space of 3-Chords

> A surprising proportion of mathematicians are accomplished musicians. Is it because music and mathematics share patterns that are beautiful?
>
> **Martin Gardner** (American Mathematics and Science Writer; 1914 - 2010)

7.1 Triadic Space

You successfully figured out the space of 2-chords. Here is a bigger challenge for you: Go to ChordGeometries and click under Geometries on Triadic Space. Why does the space of 3-chords look like a triangular prism? See Figure 7.1 for a screenshot of the triadic space.

Before we start our investigation, go to Tymoczko's website `http://music.princeton.edu/~dmitri/ChordGeometries.html` and play the video clip *Chopin through 4-dimensional space*. Isn't it beautiful? Tymoczko showed that a composer usually stays within a range of some chords in triadic space and even found composer-specific patterns in the chord progressions. The composers are probably unaware of these patterns but we can now see at least one aspect that makes their music *special*! Let's see how his program works:

1. Play with different 3-chords and see where the points appear in the prism. Can you see any patterns? For instance, where are the major chords (interval pattern[1] of 4 3)? Where are all the unison chords (all notes the same)? Explain all the patterns you find in detail.

First we need to understand how we can draw something 3-dimensional. Every chord corresponds now to 3 numbers, for instance C-major, $C\ E\ G$, equals (0 4 7). If you have Zome pieces (`http://www.zometool.com`) available, you can use the long blue pieces to make a big cube. Decide which vertex (corner) should be the origin and label the 3 edges emanating from the origin with x, y and z. Make sure that they follow the **right hand rule**: if the thumb points in the direction of the x-axis then the index funger points into y-direction and the middle finger into z-direction. This is the typical mathematical convention for 3-dimensional space. See Figure 7.2 for an example of the point (2 1 1) in 3-dimensional space.

[1] The **interval pattern** records the distances between notes, measured in half notes. For instance the C-major chord, C E G, has 4 half steps between C and G and 3 half steps between E and G.

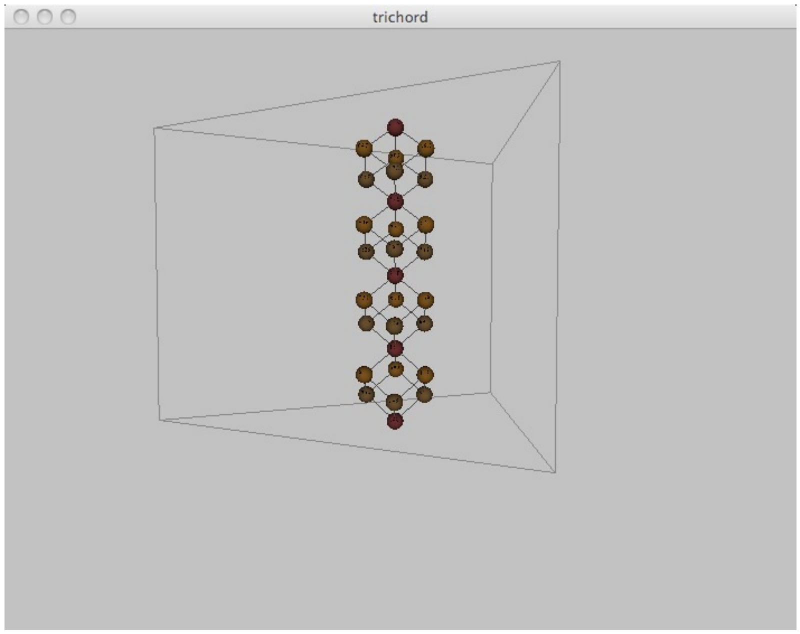


Figure 7.1: Triadic Space

Our goal is to find out how much of the 3 dimensional space we need so that each chord is represented by exactly one point.

The first step that comes to mind is to ignore the repetition of notes on the axes. That means we are just looking at one of the Zome boxes from before.

2. Imagine you walk inside that box (following a sequence of chords through space), if you walk out the left side of the box, where do you enter the box again? What if you would fly through the ceiling: where would you enter the box again?

The space you are looking at is called a ***3-dimensional torus***. As with the (2-dimensional) torus that we saw before, we would like to glue some of its sides, but unfortunately we would need 4 dimensions to picture the result. Go to `http://www.geometrygames.org/` and download the free program Curves Spaces. Choose the basic 3-dimensional torus application to visualize what the inside of a 3-dimensional torus looks like.

This is great! Unfortunately is doesn't look like the triangular prism yet that we see in Chord-Geometries. What's happening? As in the 2-dimensional case we have to consider that number triplets like (0 4 7), (4 0 7), (7 4 0) and (0 7 4) represent the same chord. We will consider first just the two triplets (0 4 7) and (4 0 7).

3. How can you change the first one to get the second one?

4. Now take an arbitrary triplet (x y z). If you change it in the same way, what do you get?

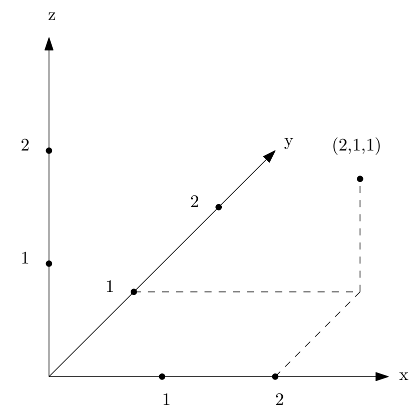


Figure 7.2: Point (2 1 1) in 3-dimensional Space

5. If we forget for a moment the third coordinate, you can picture the change of numbers in the pairs (0 4) and (4 0) as a reflection at a diagonal. Draw a picture to convince yourself of this.

In the 3-dimensional case, there is a plane inside your box, such that reflecting (0 4 7) at that plane will give you (4 0 7). The same will work for any point (x y z), giving (y x z).

6. Can you picture the plane inside your box? Use more of the Zome tools for diagonals (yellow pieces) and some paper and tape to show that plane.

7. Find the plane of reflection for the other change (0 4 7) $\leftrightarrow$ (7 4 0).

8. Find the plane of reflection for the other change (0 4 7) $\leftrightarrow$ (0 7 4).

Can you see all three planes now in your box? It helps if you also show some of the boxes next to your original box (even though all boxes are really the same). Then turn your box so that you look down one of the diagonals and see if you can detect a triangular tube beween the planes. This is one step closer to the triangular prism, since can see now where the triangle comes from!

9. Which number triplets lie on the main diagonal (going through the origin)?

10. Label some of the triplets on the diagonals in your boxes. Can you see why it is enough to just take one piece of the triangular tube?

Congratulations, you now understand the space of 3-chords!

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