Mathematics, Music, and the Guitar

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Project Theme

My project focuses on mathematics, guitars, and music theory, and centers around the following application: what if you were in a junkyard and found an acoustic guitar without strings or frets? Could you salvage it with some math and a little physics? For the purposes of simplicity and concreteness, we will use my Eastman AC520 acoustic guitar as an example when fitting.

Introduction

The guitar is one of the most incredible instruments constructed by mankind. It can produce some of the most complex, intricate music imaginable, and yet it is also accessible to even the youngest of learners. In appreciation of this instrument and musical theory, as well as the field of mathematics, we will here study some of the complex interplay between the subjects. Our subjects, among other things:

- *the relationship between frequency of sound waves and the pitch of musical notes*
- *the body of a guitar as an acoustic resonator*
- *Pythagorean tuning & and the diatonic scale*
- *the Western chromatic scale as a group of 12 elements closed under transposition in a specific system of tuning*
- *how to place frets on a guitar using either a mathematical idea or an equation from physics.*

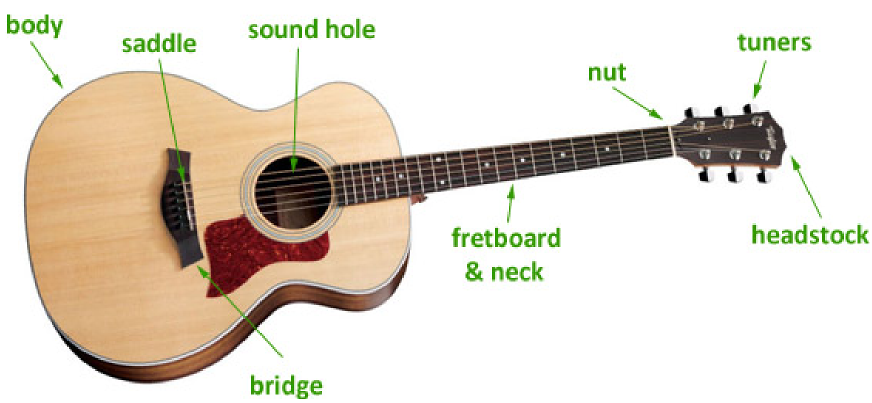


Figure 1: The major parts of a traditional acoustic guitar

1) How to calculate the volume of a guitar body, & then its frequency

The body of a guitar behaves almost identically to what is called a Helmholtz resonator, which is a rigid cavity with a tube feeding into it. When pressure is put on the air at the top of the tube, the pressure in the cavity increases and air is shot back out of the tube. This process repeats as the pressure inside and outside of the tube & cavity increase and decrease, and the result is an oscillating sound wave. An example of a Helmholtz resonator is a glass bottle; when you blow on the top of the bottle, a noise with a certain frequency comes out. Different bottles emit sounds of different frequencies, but why? With some simple physics and differential equations, it can be shown that Helmholtz resonators satisfy the following equation:

$$f = \frac{c}{2\pi}\sqrt{\frac{S}{VL}}$$

where c is the speed of sound, S is the surface area of the opening of the tube, V is the volume of the cavity, and L is the length of the tube. It should be noted that we must also compensate for what is called "end effect," which is a property of the wave travel out of the tube; the air travels a little farther than the length of the tube, and therefore L must be adjusted accordingly. In addition, we must emphasize that a guitar behaves *almost* identically to such a resonator. Because the walls of a guitar body are designed to vibrate and thus are not actually rigid, the calculations will have a certain small percentage of error. For the purposes of guitar manufacturing, though, this error is not a deterrent!

To see the formula in action, let's look at my guitar.

We need to calculate the volume of the body, the radius of the sound hole, the area of the sound hole (treating it like a flat circle), and to make sure we adjust for "end effect".

Let us calculate the volume first. Because the guitar is an irregular shape, we will need approximation methods. In my calculations, I used two different methods, each of which produced surprisingly similar results. The first was to consider the guitar to be two adjacent, large rectangular boxes (where the width is vertical, length is horizontal). The length of each box would just be half the length of the guitar body; the width would be halfway between the width of the smaller part of the body (the left half of the body) and that of the larger part (the right half of the body); the depth would be slightly different for each box. See the picture below.

The middle line is the line splitting the guitar into left and right "boxes." Note how the width of

this line is a little shorter than the width of the right part of the body, and a little longer than that of the left. The two average out to make this a decent width approximation for each box when the two are summed together. For the height of each box, I just took what seemed an "average" height; my first approximation might thus be more crude than a more refined approach. My final measurements were: length and width for both boxes - 10" and 13.5", respectively, and height - 4.125" for the left half and 4.75" for the right. For the sound hole, I measured a radius of 1.9375".

Now, our formula requires c the speed of sound, V the volume of our resonator, S the area of our sound hole (or the 'tube' into the resonator), and L the length of the tube. Since $c = 343\frac{m}{s}$, we will convert all of our measurements into meters and multiply our radius r of the sound hole by 1.7 to account for end effect (and thus get an appropriate L). Our calculation then becomes:

$$
\begin{aligned}
f &= \frac{c}{2\pi}\sqrt{\frac{S}{VL}} \\
&= \frac{343\,m/s}{2\pi}\sqrt{\frac{.00752\,m^2}{(.019294\,m^3)(.083176\,m)}} \\
&= 118.17\,\frac{1}{s} = 118.17\,Hz \approxeq A_2^\sharp
\end{aligned}
$$

Firstly, observe how the units cancelled perfectly, giving us an answer in hertz. Secondly, traditional acoustic guitars are often tuned to 110 Hz, so our formula and approximations were pretty accurate! They put my acoustic about in the range of an $A_2^\sharp$. Could we approximate the guitar's frequency more accurately, though?

A second method I attempted was to reconsider the guitar's shape as two cylinders; intuitively, a guitar does look like two different ellipses, so circles might approximate the surface area well.

The two tiny strips of paper in the middle of each half of the guitar are what I estimated would be the center of each circle. The strips on the edges of the guitar are where I took the various measurements from their nearest centers. After each measurement was taken, I calculated two arithmetic means to get an estimate for what the radius of each circle would be. Afterward, I measured heights at 4 & 5 different points on each half of the guitar, respectively, and then took the arithmetic mean of *these* to get a more refined figure for height than I measured in my first attempt. For left circle A and right circle B, my new calculations were:

$$h(A) = (4.625 + 4.4375 + 4.375 + 4.0625 + 3.9375) \div 5 = 4.2875''$$

$$r(A) = 45.25 \div 8 = 5.6525'' \Rightarrow S(A) = 100.376\, in^2 \Rightarrow V(A) = 430.934\, in^3$$

$$h(B) = 19 \div 4 = 4.75''$$

$$r(B) = 69 \div 10 = 6.9\text{''} \Rightarrow S(B) = 149.571\, in^2 \Rightarrow V(B) = 710.463\, in^3$$

$$\Rightarrow V(guitar) = V(A) + V(B) = 1141.397\, in^3$$

The S and L remain the same as the original calculation, but the V has changed, and we will also avoid inexact unit conversions by switching directions and converting $c = 343\frac{m}{s}$ into $\frac{in}{s}$ to get

$$f = \frac{13582.8\,\frac{in}{s}}{2\pi}\sqrt{\frac{11.793\, in^2}{(1141.397\, in^3)(3.29375\, in)}}$$

$$= 121.1\, Hz$$

This guitar therefore more likely has a frequency close to 121.1 hertz, and rings out around an $A_2^\sharp$ (which we will help understand more later!).
Therefore, with the concept and equation of a Helmholtz resonator and some basic or more complex approximation methods, we could determine the frequency range of any old guitar whether it had strings or not. We would probably want to put on some strings & frets, though, so let's look at some of the mathematics that will help us understand how to go about doing so.

2) Frequency & music theory

All of music theory - or at the very least, music theory in the West - has been intricately tied to mathematics for thousands of years. We will here detail some of these origins, discussing frequency, diatonic & chromatic scales, and various tuning systems from Pythagoras' day to modern times.
Sound travels to our ears in waves, or certain periodic oscillations or vibrations. This is relatively easy to picture with a string instrument, because a string with the right amount of tension visibly vibrates up and down, much like a rubber band stretched incredibly tight. Some string vibrations are harsh, and strike our ears as what we call "noise." Some sound waves, though, strike our ears as music, and it is these waves we concern ourselves with. Sound waves, especially ones we classify as music, are by nature periodic - i.e. they repeat themselves - and have a few major characteristics. One of these is the *period*: how long it takes the wave to repeat itself. For example, the period of $f(x) = \sin(x)$ is 2π, because $\sin(x) = \sin(x + 2\pi)$ for all x. The inverse of the period, on the other hand, is the *frequency.* If the period is measured in seconds, the frequency is thus measured in units $\frac{1}{s}$, which is the unit of the *hertz.* The frequency can thus be thought of as how many cycles the wave makes per unit time (which, for our purposes, will always be $s =$ seconds). Therefore, the more cycles the wave makes per second, the higher the frequency will be.
Thus far, we know that sound waves travel to our ears in waves, and that the more these waves cycle through per second, the higher their frequency will be by definition. Intuitively, we also know that high frequencies associate with high pitched sounds; the tighter a rubber band is, the faster it vibrates, and the higher pitched sound it makes. Indeed, our ears actually perceive pitch directly as a result of frequency; the two are, for all intents and purposes, the same to our ears. What pitches sound pleasant to our ears, though? In other words, what frequencies do we want the sound waves of our guitar strings to have? More importantly, what combinations of two notes - two waves with certain frequencies - sound *consonant*, or pleasant together?

Let's start with the basics: there are indeed some sound waves that sound better than others. A wave with a frequency of 440 Hz, for example, is what is now referred to as A4, and is the reference note for many major musical instruments. C4, or the "middle C" of the traditional 88-key piano, has a frequency of around 261 Hz. We could continue naming notes by trial-and-error with various frequencies, but could we not do this indefinitely? We must consider what notes in a relatively small range sound pleasant together.
Any two notes played at the same time have two different frequencies, and the ratio of their frequencies is called an *interval.* The most basic interval is the *unison*, which is a note played with itself, with ratio 1:1. Clearly, any musical note will sound consonant with itself; two C notes played at the same time will in no way directly aesthetically conflict with each other. The second most natural interval is the *octave.* An octave is two notes that have a frequency ratio of 2:1. Thus, a *C4*-*C5* will be an octave; both notes are C, but *C5* has double the frequency of *C4*. If we wanted to create a scale of notes between *C4* and *C5*, what notes, or frequencies, would we choose, though?

Throughout history, *many* scales have been created & employed in the design of musical instruments. Some of these, in fact, have largely been developed by heuristics; some notes simply sound good together, and thus can be turned into a scale. Among the earliest well-known tuning systems based on mathematics, though, was that of the Greek mathematician Pythagoras, better known for his theorem concerning right triangles, of course. Legend has it that Pythagoras actually got the idea for his tuning system by hearing blacksmiths hammer metal and learning that the hammers' masses were in simple integer ratios. Thus, his idea - what is now called *Pythagorean tuning* - was that *all consonant sounds have frequencies in simple small integer ratios with one another.*

Consider the note C at any frequency, and note C' one octave above C. Pythagoras theorized that every consonant note between C and C' could be arrived at by multiplying and dividing by 2 & 3, with appropriate scaling when necessary. The first note formed within the scale would be the most basic possible ratio of C - namely, $(\frac{3}{2})C$, a note with frequency ratio of 3:2 with the base note C. This note would become G. Pythagoras discovered that this interval of 3:2 was actually incredibly consonant - so consonant in fact that this interval is now called a *perfect fifth.* Therefore a new ratio, besides the octave 2:1, was discovered important. He proceeded to scale C', the high C, *down* by a perfect fifth, arriving at a note with ratio $\frac{2}{\frac{3}{2}}$: $1 = 4$:3 with C. This note would become F, and the interval 4:3 would be named a *perfect fourth.* With F & G now created, we can consider *their* interval $\frac{3}{2} : \frac{4}{3} = 9 : 8$. This interval would become the *major second*, or *whole tone.* In this way, Pythagoras continued and constructed the following scale of 8 notes, or a *diatonic scale*:

$$\begin{array}{cccccccc} C & D & E & F & G & A & B & C' \\ \frac{1}{1} & \frac{9}{8} & \frac{81}{64} & \frac{4}{3} & \frac{3}{2} & \frac{27}{16} & \frac{243}{128} & \frac{2}{1} \end{array}$$

where the number below each note is its frequency ratio with C.

Note that each number is, beautifully, of the form $\frac{3^m}{2^n}$, for positive integer values of m, n. Similarly, notice that we had to make use of our octave: in finding the note one perfect fifth, or 3:2 above $G = \frac{3}{2}$, we got $\frac{3}{2} * \frac{3}{2} = \frac{9}{4}$, which is greater than $\frac{2}{1}$ and therefore outside our octave. We utilized the fact that a note is consonant with any multiple of itself by a positive or negative power of 2; thus $\frac{9}{4} * \frac{1}{2} = \frac{9}{8} = D$.

Pythagoras' system would be perfect for our guitar, right? It appears we have a cyclic group of notes by the definition of our frequencies - every note has frequency that is a simple positive rational multiple of the other. This scale, though, is for the seemingly arbitrary note C - what if we wanted to change, or *transpose*, a piece of music or our instrument to the key of G? We would essentially want to translate every note by the "distance" 3:2, the interval of C-G. Let us see what happens if we transpose each note upward by this perfect fifth by multiplying each fraction, or interval, by $\frac{3}{2}$:

$$C = \frac{1}{1} \to \frac{3}{2} = G$$

$$D = \frac{9}{8} \to \frac{27}{16} = A$$

$$E = \frac{81}{64} \to \frac{243}{128} = B$$

$$F = \frac{4}{3} \to \frac{12}{6} = \frac{2}{1} = C'$$

$$G = \frac{3}{2} \to \frac{9}{4} \equiv \frac{9}{8} = D$$
$$A = \frac{27}{16} \to \frac{81}{32} \equiv \frac{81}{64} = E$$
$$B = \frac{243}{128} \to \frac{729}{256} \equiv \frac{729}{512} = \frac{3^6}{2^9} = ?$$
$$C' = \frac{2}{1} \to \frac{3}{1} = 2 * \frac{3}{2} = G'$$

Everything looks good in that each note, when transposed, went right back to another note in the scale - *except for* B. The resulting fraction, $\frac{729}{512}$, is not in our original list of eight notes, and cannot be reduced by the uniqueness of prime factorization. Therefore, the Pythagorean scale of C is not a closed group under transposition. In fact, the interval $\frac{729}{512}$ produced a *new* note, which would come to be known as $F^\sharp$, or F *sharp*.

By transposing from the key of C to the key of G using Pythagoras' idea of integer ratios, we encountered a problem - we found a new note. If this new note was the only 'problem note', we should theoretically be able to continue our transposition smoothly. Unfortunately for Pythagoras, however, transposition using integer ratios is actually very problematic. If we continued to transpose by perfect fifths, or the fraction $\frac{3}{2}$, we will produce another sharp for every transposition; therefore two transpositions by a perfect fifth, or a transposition by $\frac{3}{2} \times \frac{3}{2} = \frac{9}{4} = 2 * \frac{9}{8}$ into the key of D, will result in *two* sharps, and so on and so forth. After six such transpositions, we will have the key of $(\frac{3}{2})^6 = F^{\#}$ with six sharps.
One might imagine that the problem lie in our transposing by fifths, rather than fourths; however, transposing by fourths actually causes similar problems. Suppose we transposed C by a perfect fourth to the key of F:

$$C = \frac{1}{1} \to \frac{4}{3} = F$$
$$D = \frac{9}{8} \to \frac{3}{2} = G$$
$$E = \frac{81}{64} \to \frac{27}{16} = A$$
$$F = \frac{4}{3} \to \frac{16}{9} =?$$
$$G = \frac{3}{2} \to \frac{12}{6} \equiv \frac{2}{1} = C$$
$$A = \frac{27}{16} \to \frac{9}{4} \equiv \frac{9}{8} = D$$
$$B = \frac{243}{128} \to \frac{81}{32} \equiv \frac{81}{64} = E$$
$$C' = \frac{2}{1} \to \frac{8}{3} = 2 * \frac{4}{3} = F'$$

Again, we have a problem with the interval $\frac{16}{9} = \frac{2^4}{3^2}$. This note is what would become B^b. Similarly to transposing by fifths, the Pythagorean scale produces a new flat for each transposition up by a perfect fourth until six flats have been discovered in the key of $G^\flat$.

Now, what if we transposed by a perfect fifth or a perfect fourth *more* than six times? We have a diatonic scale of 8 notes, and by our method of transposition to *non*-sharps or flats, any root note and its octave cannot possibly transpose to a sharp or flat; therefore we should never have more than (8-2) = 6 flats or sharps. However, we have by our previous twelve transpositions (starting with C, we went up by fifths six times and up by fourths six times) and twelve new notes. From basic knowledge of a piano, though, we know that, for example, $F^\sharp$ and $G^\flat$ are the same thing - why are they the same, though? Is this mathematically justified?
Let's consider what happens to C when we do transpose into flats and sharps, say $G^\flat$ by perfect fourths and $F^\sharp$ by perfect fifths. We already know from earlier that C-$F^\sharp$ is an interval of $\frac{3^6}{2^9}$, and it can be shown that C-G^b is an interval of $\frac{2^{10}}{3^6}$. If we were to set these equal to each other, which by assumption we *should* be able to do, we would get

$$\frac{3^6}{2^9} = \frac{2^{10}}{3^6} \Rightarrow \frac{3^{12}}{2^{19}} = 1$$

By the uniqueness of prime factorization, we know that this cannot possibly be true. Is it even close, though? It turns out that the ratio $\frac{3^{12}}{2^{19}}$ of the two Cs after transposition to $F^\sharp$ and $G^\flat$ is actually 1.013643... which is incredibly close to 1. This error is referred to as the *Pythagorean comma*, which has been accepted as a small enough error to continue using Pythagoras' ideas in creating a better musical scale. In other terms, the Pythagorean comma has been accepted and utilized because $\{m = 12, n = 19\}$ is the solution set with the smallest integers that allow $\frac{3^m}{2^n} \approx 1$ and remain practical. In fact, if we rewrite our desired equation representing fifths that generate octaves as

$$(\frac{3}{2})^m = 2^q$$

we can then rearrange to get

$$\frac{3^m}{2^{m+q}} = 1$$

where m is the number of notes in the scale and q is the number of semitones spanned by the perfect fifth. With our above approximation, we thus get a scale with 12 musical notes and (19 - 12) = 7 semitones spanned by the perfect fifth! (The next smaller integer solution set is $m = 53, q = 31$, meaning we would have to have 53 notes in our scale!)
Indeed, this notion is critical for the development of the tuning system many instruments use today. Though Pythagorean tuning is not the ideal musical scale nor the perfect system for fretting our guitar due to its production of flats and sharps, its idea of an interval - in Pythagoras' case, a perfect fifth - that can generate the entire scale would serve as the conceptual basis of the system of tuning that we will use, called *equal temperament*. It also, as we will see later, produces a circle that is incredibly useful for virtually any musician.

3) Equal Temperament

The system of tuning that has proven to provide both acceptable harmonic accuracy and freedom for easy transposition is called *equal temperament*. Equal temperament arose from a search for an interval that would preserve the ratio $\frac{3}{2}$ of the perfect fifth *and* be completely cyclic, i.e. satisfy the equations

$$x * x * x * x * x * x * x = \frac{3}{2}, \quad and$$

$$x^{12} = 2 \quad (\bigstar)$$

The first equation is a mathematical representation of a scale as a geometric series that preserves the perfect fifth. In other words, we want an interval such that a transposition by that interval seven times - seven semitones, which we know we desire by the accuracy and practicality of the Pythagorean comma - is exactly a perfect fifth, which, in perfect harmony, has a ratio 3:2. The second equation, on the other hand, represents the fact that our geometric scale is for a scale of 12 notes, also due to the Pythagorean comma. A 12 note scale is actually called a *chromatic scale*. In fact, the scale

$$\{C, C^\sharp/D^\flat, D, D^\sharp/E^\flat, E, F, F^\sharp/G^\flat, G, G^\sharp/A^\flat, A, A^\sharp/B^\flat, B\}$$

is known as the *Western chromatic scale*.
Now, we could just solve both equations for x, but it's clear that the two solutions will not be equal; which, then, is more important, $\sqrt[7]{\frac{3}{2}}$ or $\sqrt[12]{2}$?
It would help to consider what would make transposition easiest - after all, this is the primary problem with Pythagorean tuning. Transposition, as we have seen, is a matter of shifting every note by a fixed distance, or interval. The problem in Pythagorean tuning was that some notes would be shifted to new notes. This problem arose from the fact that the "distances," or intervals between the notes of the diatonic scale, were not all equal. Some, like C-D, were $\frac{9}{8}$, while others, like D-E, were $\frac{10}{9}$. While both slightly greater than 1, these intervals are not equal, nor can one be multiplied or divided by powers of 2 and 3 to get the other.
Within equal temperament, we thus want some form of distance to be preserved - if all the notes are equally spaced, then transposition will not lead to any unexpected notes. The most basic method would be to consider the octave linearly. If we divided the linear octave into 12 equal intervals, then the distance between each pair of consecutive notes would be $1 * \frac{1}{12} = \frac{1}{12}$ and each interval would be $(\frac{1}{12} + 1) : 1 = \frac{13}{12}$. However, we know that the perfect fifth should be very close to $\frac{3}{2}$, and this linear scale produces a fifth of $\frac{(13^7)}{(12^7)} \approx 1.75$, which is nowhere close to what we want. Thus, a linear scale will not work.
Looking back to equation $(\bigstar)$, though, we could consider how to use the solution $x = \sqrt[12]{2}$ to equally space our notes. It is clear that $f(x) = 2^{\frac{x}{12}}$ is an exponential function, and thus does not increase linearly. However, on a *logarithmic scale of base 2*, $f(x) = 2^{\frac{x}{12}}$ is indeed linear, as for $m, n \in \mathbb{N}$ we have constant slope

$$\frac{log_2(2^{\frac{m}{12}}) - log_2(2^{\frac{n}{12}})}{m-n}$$

$$= \frac{\frac{1}{12}(m-n)}{m-n} = \frac{1}{12}$$

Therefore, if we use an interval of length $\sqrt[12]{2} = 2^{\frac{1}{12}}$, our octave will be an interval of length $(\sqrt[12]{2})^{12} = 2$, which is exactly what we need to be true, and our notes will all be linearly separated by a distance of $\frac{1}{12}$ on a logarithmic scale. Now all we have to check is if seven semitones of length $2^{\frac{1}{12}}$ is close to a perfect fifth:

$$(\sqrt[12]{2})^7 = 2^{\frac{7}{12}} \approx 1.4983$$

This is incredibly close to $\frac{3}{2} = 1.5$, so it appears this system of tuning is perfect for what we want. This system is in fact known as *equal temperament*, and we can use this to place the frets on a guitar, because we now know exactly what ratio we want our notes, and as a result our fret lengths,

to have with each other.
Note: Music theorists actually came up with a unit of distance for notes in equal temperament called a *cent*. The distance between two consecutive notes is 100 cents, or $1200 * \frac{1}{12} = 1200 * \log_2(2^{\frac{1}{12}})$. Therefore, for any two notes with frequencies of α, β, one can find the interval between α & β in cents by calculating $d(x, y) = 1200 * \log_2(\frac{\beta}{\alpha})$.

4) Two ways to place frets on a guitar

To put frets on a guitar neck, we can use two methods. The first is using equal temperament, which we discovered using musical theory. The other relates to physics and the linear density and tension of a given string.
To use equal temperament to fret our guitar, we must first consider the inverse relationship between string length and sound frequency. If we have a string of length L, it will ring out with a certain frequency. If we place our finger halfway down the string, and pluck on either side of it, the string will vibrate with *twice* the frequency since the wave has *half* the distance to travel. In other words, the length of string and frequency of the sound waves is inverse; the shorter the string, the higher the frequency and thus the pitch of the note.

Therefore, if we want our first note to have a ratio of $\sqrt[12]{2} : 1$ with, or, equivalently, a frequency $\sqrt[12]{2}$ times larger than the frequency of the open note (the note sounded by plucking the string without pressing down on the string anywhere), we will need the *length* of the string we pluck to be $\frac{1}{\sqrt[12]{2}}$ times L the length of the whole string (from the nut to the bridge). Hence, we want to put our first fret so that, when we push down with our finger, the length of string that rings out is $\frac{1}{\sqrt[12]{2}} * L$, which implies that the fret should be placed $(1 - \frac{1}{\sqrt[12]{2}}) * L$ away from the nut.

Let's test this formula on my guitar. The length from the nut to the bridge is $25\frac{5}{8}$". We estimate that we want the first fret to be placed

$$(1 - \frac{1}{\sqrt[12]{2}}) * 25\frac{5}{8}\text{"} \approxeq 1.438''$$

away from the nut. When actually measured, the length of the fret was approximately $1\frac{7}{16}$" = 1.4375". The formula is indeed satisfactory, then.

Now, we could continue in this manner, using the length of string from the end of the fret to the bridge as the new length of the string each time. One important feature of this process is that the frets will become closer and closer together; it is common knowledge that the frets are very close together close to the bridge, but why?
The reason is that $\sqrt[12]{2} \approx 1.059 > 1 \Rightarrow 0 < \frac{1}{\sqrt[12]{2}} < 1 \Rightarrow 0 < 1 - \frac{1}{\sqrt[12]{2}} < 1$. Since the length of a semitone is greater than 1, the length of string - which has an inverse relationship with frequency - needed to ring out a note one semitone higher than a previous note is *less* than 100% of the remaining string. In fact, we have only needed to use $\frac{1}{\sqrt[12]{2}}$ of the remaining string for each fret we put down. As we do this infinitely, the frets will become infinitesimally small. We could also state this as follows:

$$\lim_{n\to\infty}(1 - \frac{1}{\sqrt[12]{2}})^n = 0$$

since

$$\lim_{n\to\infty} r^n = 0$$

for any $r \in [-1, 1]$. How many frets we have placed is n, so as we place more and more frets, the length of the frets get shorter and shorter.

We could also place our frets without using equal temperament at all, but rather using the following formula:

$$L = \frac{\sqrt{\frac{T}{\rho}}}{2f}$$

where T is the tension of the string, ρ is the linear density of the string, and f is the frequency. Many guitar makers, such as Martin Guitars, arguably the most renowned maker of acoustic guitars in the world, provides data about the gage (diameter) of their strings as well as the recommended tension[1]. If we wanted to put, say, medium-light Marquis strings on our guitar, we would know that our low E-string would have a diameter of 0.056" and a recommended tension is 181.1 Newtons. For our junkyard guitar, though, we would already have a fixed length and we would buy strings with a fixed gauge and linear density; therefore, we would choose the frequency we want and then tune our strings by increasing the tension accordingly!

Let's run through a quick calculation. Let's say we want a low E, and our string has a diameter (*gage*) of .036". Our guitar has length 25.625" $\approxeq$ .647 m from the nut to the bridge. Using figures for the mass of strings a project done at Louisiana Tech University we calculate that the linear density of our low E-string is the mass of the string minus the mass of the ball at the end (used to keep the string in place) divided by the length of the string, or

$$\begin{aligned}
\rho &= \frac{mass_{string} - mass_{ball}}{L} \\
&= \frac{4.4\,g\; - \;0.2\,g}{25.625''} \\
&= \frac{.0042\,kg}{.647\,m} \\
&= .00649\frac{kg}{m}
\end{aligned}$$

For our string to ring out open as an $E_2 \approxeq 82.41Hz$, we need tension

$$L = \frac{\sqrt{\frac{T}{\rho}}}{2f}$$

$$\Rightarrow T = 4\rho L^2 f^2$$

$$\Rightarrow T = 4(.00649\,\frac{kg}{m})(.647\,m^2)(82.41\,\frac{1}{s}^2)$$

$$\Rightarrow T = 73.8\frac{kg \cdot m}{s^2} = 73.8\,N$$

Given the tension of the open E-string, we can use the known frequencies of notes of the Western chromatic scale and the above equation to figure out L, the length away from the bridge of the placement of our frets.

For one example, let's say we want to place our first fret on our guitar. For the E-string, we would want the first fret to play the next note in the chromatic scale (from E_2), which is F_2 that has a frequency of 87.31 Hz. We use our equation to calculate:

$$L = \frac{\sqrt{\frac{T}{\rho}}}{2f}$$

$$= \frac{\sqrt{\frac{73.8}{.00649}}}{2(87.31)}$$

$$= .6107\, m = 24.183\, in$$

Therefore, we would want to place our fret $25.625 - 24.183 = 1.442$ inches away from the nut. Note that, using our other method, we calculated a fret distance of 1.438 inches. The error that resulted between using the two formulas was only 0.3%!

<u>RECAP SO FAR</u>

Given a guitar with no frets or strings, we now know how to determine its loudness, how to place frets on it using either of two different methods, and the system of tuning (equal temperament) to which guitars are tuned. The last thing we will discuss is a ramification of Pythagoras' ideas in modern music theory - the circle of fifths - and how it informs the traditional tuning of six string guitars.

<u>5) The Circle of Fifths (or Fourths)</u>

Pythagoras' search for a cyclic group of notes that would be closed under transposition was, with his notion of integer ratios, technically a failure. However, demonstrating *why* Pythagorean tuning is not closed under transposition resulted in the invention of the Western chromatic scale and a relationship between the number of sharps or flats in a key and how many fourths or fifths the root note of a key is away from a C.

After the discovery of the 5 flats and sharps (there were 10, technically, but $F^\sharp \approxeq G^\flat$, etc. reduced the 10 to 5 pairs of approximately consonant notes) not in the C-D-E-F-G-A-B-C diatonic scale, the invention of equal temperament demonstrated that the perfect fourth represented 5 linear steps along the logarithmic scale and the perfect fifth 7 steps along the same scale.

These two numbers, 5 & 7, are incredibly important considering there are specifically 12 notes in the Western chromatic scale. Observe:

1) 5 + 7 = 12.

Hence, 12 -7 = 5 implies that a note, say C_n, scaled *down* by a perfect *fifth* is equivalent to C_{n-1} (an octave below C_n) scaled *up* by a perfect *fourth*.

Similarly, 12 - 5 = 7 gives that transposing a note down by a perfect fourth is equivalent to transposing the same note an octave below up by a perfect fifth.

2) Because $7 \nmid 12$, we have that $gcf(7, 12) = 1$.

Hence, the chromatic scale, which (because of equal temperament) can be written as $\{C, C^\sharp/D^\flat, D, D^\sharp/E^\flat, E, F, F^\sharp/G^\flat, G, G^\sharp/A^\flat, A, A^\sharp/B^\flat, B\} := \mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ can be generated by multiples of 7:

$$\{7k(mod12) : k = \{0, 1, 2, ...\}\} = \{7 \cdot 0, 7 \cdot 1, 7 \cdot 2, ...\}$$

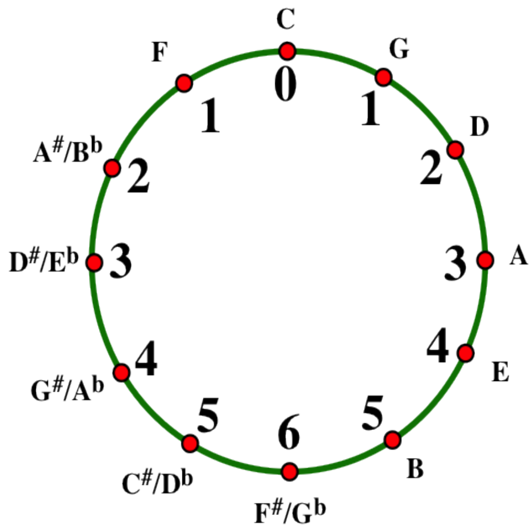
$$= \{0, 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, ...\}$$

$$= \{0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, ...\}$$

which we rewrite back into notes as

$$\{C, G, D, A, E, B, F^{\sharp}/G^{\flat}, C^{\sharp}/D^{\flat}, G^{\sharp}/A^{\flat}, D^{\sharp}/E^{\flat}, A^{\sharp}/B^{\flat}, F\}$$

This gives rise to the following, called the circle of fifths:



Starting with C and rotating clockwise, each note is a perfect fifth up from the previous note, i.e. G is a perfect fifth (7 semitones) up from C, B is a perfect fifth up from E, etc. By fact (1) above, though, we could also go counter-clockwise from C, in which case we would be moving *down* by *perfect fourths.* Thus, backwards, the circle of fifths is a circle of fourths. Now, one might notice that the numbers from the circle of fifths do not go from 0→11, but $0 \to 6 \to 0$. Think back to the discovery of the flats and sharps: by transposing the diatonic scale C up by a perfect fifth, we got the key of G, which had a new note, $F^{\sharp}$. Here, then, in the diagram, the number by each note represents the number of sharps (or flats) in that key! Mathematically, then, one can label the chromatic scale (starting at C) 0 through 11, and for a note with label k, the number of sharps or flats in that key will be:

$$7k(mod 12) \text{ sharps}, \text{ if } 7k(mod 12) \leq 6$$

$$[7k(mod 12) - 2(7k(mod 12) - 6)] \text{ flats}, \text{ if } 7k(mod 12) > 6$$

For example, since $A^{\sharp}/B^{\flat}$ is 3 semitones away from C, we calculate

$$10 \cdot 3(mod 12) = 8 > 6$$

$$\Rightarrow B^{\flat}\text{will have } [8 - 2(8 - 6)] = 4 \text{ flats}$$

The *standard tuning* for a guitar is E-A-D-G-b-e, from low to high, and is designed to make playing chords as easy as possible. From our knowledge of the circle of fifths, and the corresponding circle of fourths counterclockwise, we see that $E - A$, $A - D$, $D - G$, and $b - e$ are all perfect fourths, or five semitones away from each other. $G - b$, on the other hand, is four semitones, which is called a *major third*. Now, we know that the perfect fourth was the *second* perfect interval developed by Pythagoras; the perfect fifth was discovered first. Why, then is the guitar tuned using primarily fourths?

Consider what it would be like to play the guitar. What would be the biggest limitation to playing incredibly intricate music? By far the most significant handicap is the number of fingers we have available to press down strings on the fretboard: *four*. In addition, our four fingers also can only stretch so far. Generally, the largest distance our four fingers can possibly cover while our thumb is wrapped the neck of a guitar is around 5.5 or 6 inches. Now, looking back to what the length of the first fret was on my Eastman guitar, and then calculating what the length of the next three frets would be:

$$\text{Length of first fret} = 1.438\text{''}$$

$$\text{Length of second fret} = (1 - \frac{1}{\sqrt[12]{2}}) * (25.625\text{''} - 1.438\text{''}) = 1.358\text{''}$$

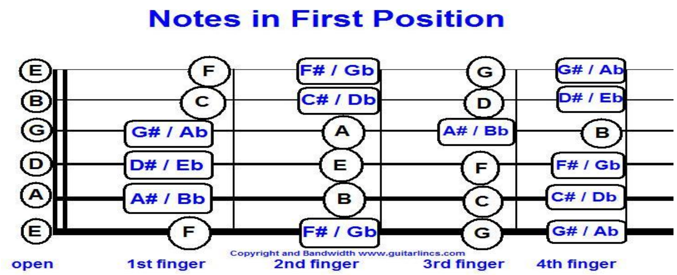
$$\text{Length of third fret} = (1 - \frac{1}{\sqrt[12]{2}}) * (25.625\text{''} - 1.438\text{''} - 1.358\text{''}) = 1.281\text{''}$$

$$\text{Length of fourth fret} = (1 - \frac{1}{\sqrt[12]{2}}) * (25.625\text{''} - 1.438\text{''} - 1.358\text{''} - 1.281\text{''}) = 1.209\text{''}$$

Therefore, the length of the first four frets combined is $1.438\text{''} + 1.358\text{''} + 1.281\text{''} + 1.209\text{''} = 5.286\text{''}$. It's easy to see, then, that four fingers should be able to cover these frets without *too* much of a stretch, but that five frets would be almost impossible to span.
But our frets were constructed to each represent one semitone, so another way of thinking about the span of a hand is to consider that the hand can cover at most 4 semitones, plus the open note (i.e. the string without any fingers pressed down), which is a total of 5 semitones, or exactly a perfect fourth. This is the reason why four of the five intervals between the strings of a guitar under standard tuning are perfect fourths - they guarantee that we can play all 12 notes of the Western chromatic scale, *without* having to stretch our fingers more than 4 frets. Since any chord is composed of some combination of these 12 notes, we can then play *any* chord without having to leave the first four frets, or what is called the *first position*. The figure on the top of the next page shows the notes contained in the first position.

If you start with the low E (the E at the bottom left of the diagram) and follow the path from the open note to the right, to the next open note, etc., one can trace out the chromatic scale going from E to E an octave higher (the third [D] string, second fret), then another chromatic scale to an E two octaves above, which is the open high e string. The first four frets therefore contain over two full octaves and a huge combination of possible chords.

This means that all of the best chord combinations can be found near each other (show). Now, to conclude this paper on a bit of a simpler note, let us demonstrate the significance and validity of the circle of fifths - arrived at mathematically starting with Pythagoras and ending with equal



temperament - by looking at some well known songs. These are the 7 of the 10 greatest songs of all time that involve guitars, according to Rolling Stone, and their major chord progressions. (Look at the circle of fifths. Notice anything?)
1) "Like a Rolling Stone" - Bob Dylan
- 'How does it feel?" - C F G
2) "(I Can't Get No) Satisfaction" - The Rolling Stones
- 'I can't get no satisfaction... 'Cause I try and I try and I try and I try' - E A E B7 E A
3) "Imagine" - John Lennon
- 'You may say I'm a dreamer...' - F G C Cmaj7 (E E7)
4) "What's Going On" - Marvin Gaye
- 'Talk to me, so you can see, Ohhh, what's going on, what's going on" - Em7 A13sus4 Dmaj7 Bm7
7) "Johnny B. Goode" - Chuck Berry
- Bb Eb Eb F
8) "Hey Jude" - The Beatles
- F C F C7 F Bb F C F
9) "Smells Like Teen Spirit" - Nirvana
- Fm Bb7 Gbm Db7 (a little more dissonant because no Ab!)

An interesting project done at McGill University may be of interest for those who learn or process visually. The project shows how notes and chords form a circle in one dimension and a Möbius strip in two dimensions, and even extends to three dimensions to make an interesting shape. All of this can be viewed at http://www.cs.mcgill.ca/ ethul/pub/course/comp644/project/index.html

For those who may play guitar or are starting to learn the instrument, the following link discusses a clever method for better accustoming oneself to the fretboard. In particular, it helps one to navigate the fretboard (which, for an average guitar, contains at least 126 notes) without using a book or sheet music, but rather, mathematics. http://www.guitarnoise.com/lessons/from-math-to-music/

Video

Here is an accompanying video of me with an actual guitar, where you can hear some of the sounds of and see some of the relationships (on the guitar) discussed in this project. Save or download the pdf, and the link below will be a hyperlink that goes straight to the video.
http://tinyurl.com/kxrza6v

Conclusion

Though this was only a survey of the mathematics that goes into understanding how a guitar works or constructing a musical scale or composing a beautiful song, we have here seen some of the intricate interplay between mathematics and music. Further, we could now, using some deceptively simple mathematics, know how and where to place frets on a bare guitar, as well as how to tune this guitar given any set of strings.

The project above was not designed to provide instructional material for teachers. Nonetheless, for future or current teachers who might consider using any of the above mathematics in their classroom, the following standards are a few that might be covered if the material above were expanded to create problems for students.

MCC9-12.A.SSE.3c - *Use the properties of exponents to transform expressions for exponential functions. For example the expression* 1.15^t *can be rewritten as* $(1.15^{\frac{1}{12}})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*
- Lengths of frets and octaves as $2 = (2^{\frac{1}{12}})^{12}$; determining frequencies of notes using this exponential representation and the frequency equations used in the project

MCC9-12.F.BF.5 - *Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.*
- Logarithmic scale & cents as linear distances between notes

MCC9-12.G.GMD.1-3 - *Explain volume formulas and use them to solve problems.*
- Approximation methods to calculate the volume of a guitar

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