MATHEMATICAL MODELLING OF RESONANCE IN AN OPEN- ENDED CYLINDER

A Case of Marimba Resonator

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By

Geofrey Mhagama Charles

A Dissertation Submitted in Partial Fulfilment of the Requirements for the Degree of Master of Science (Mathematical Modelling) of the University of Dar es Salaam

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CERTIFICATION

The undersigned certify that they have read and hereby recommend for acceptance by the University of Dar es Salaam a dissertation titled: ***Mathematical Modelling of Resonance in an Open- Ended Cylinder: A Case of Marimba Resonator,*** in partial fulfilment of the requirements for the degree of Master of Science (Mathematical Modelling) of the University of Dar es Salaam.

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DECLARATION

AND

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I, **Geofrey Mhagama Charles** declare that this dissertation is my own original work and that it has not been presented and will not be presented to any other University for a similar or any other degree award.

Signature____

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DEDICATION

This dissertation is dedicated to my family for their support and patience during the preparation of this dissertation. In particular, I would like to mention Prof. R. W. P. Masenge, Dr. D. J. Henwood, and Dr. N. R. Mlyuka for their help, encouragement and support during my research work

LIST OF ABBREVIATIONS

2D	Two Dimensions
Dr.	Doctor
FDM	Finite Difference Method
FD	Finite Difference
FEM	Finite Element Method
FE	Finite Element
h_r	Mesh width in the r-direction
h_z	Mesh width in the z-direction
Le	Effective length of the cylinder
Lr	Lengths of cylinder in the r-direction
Lz	Lengths of cylinder in the z-direction
NOMA	Norad Programme for Masters Studies
nr	Number of nodes in the r-direction
nz	Number of nodes in the z-direction
Prof.	Professor
PVC	Poly Vinyl Chloride
UDSM	University of Dar es Salaam

ABSTRACT

A mathematical model for analysing the natural frequency of an open-ended cylindrical resonator is formulated, simulated and validated using laboratory experiments. The Marimba was targeted for application of the findings, because it has several such resonators. Of particular interest is the influence of the length and radius of the cylinder on its natural frequency, and sensitivity of the natural frequency to small changes in either the length or the radius of the cylinder.

The mathematical model consisted of Helmholtz's partial differential equation (in cylindrical coordinates) coupled with a set of Neumann-type boundary conditions prescribed on both ends of the cylinder, and along the axis, and the surface wall of the cylinder.

The finite difference method (FDM) was used to convert the continuous differential equation into a corresponding discrete difference equation that was used to generate a system of simultaneous linear equations whose solution was the required velocity potential. The system was solved using MATLAB software. Both simulation and validation of the model, confirmed the researcher's hypotheses that the length of the cylinder has a direct influence on its natural frequency, and that the radius has little, if any.

Sensitivity analysis carried out by making small changes in the length of the cylinder near the optimal length for producing resonance revealed that, the natural frequency is very sensitive to small variations in the length of the cylinder and insensitive to variations of the radius unless the radius is unreasonably large.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER ONE

INTRODUCTION

1.1 General Introduction

Marimba is an idiophone that is sounded by striking wooden bars with mallet (Fletcher and Rossing, 1998). Marimba is defined as a pipe percussion instrument that produces sound through a vibrating air column (Nederveen, 1998). The name originates from Bantu languages where by “rimba” means a note or key, while “Ma” is a plural prefix. Generally, Marimba is equivalent to many sound keys.

A Marimba has three basic attributes, namely, the name itself, the presence of a vibrating membrane and the presence of a thin bar of wood as part of the instrument.

Figure 1.1: Picture of a Modern Marimba

Marimba resonators have been changing with time, from those made traditionally using gourds to the modern ones with metal pipes (usually brass or aluminum). Below each marimba bar hangs a resonator whose length is related to the frequency produced by the bars.

The modern Marimba (Figure 1.1) consists of a set of keys made of wooden bars, preferably rosewood, mahogany or African padouk, with metal resonator cylinders. The bars are struck with mallets to produce musical tones. Directly beneath each Marimba bar is a resonator cylinder. The resonator cylinder obeys the physical laws for standing waves in a pipe.

Fletcher and Rossing (1991) point out that, Marimba resonators (Figure 1.1) are cylindrical pipes tuned to the fundamental mode of the corresponding bars .The pipes are open at the end just below the vibrating bar as well as at the other end. Sound wave travels down the pipe and is reflected back at the other open end. Consequently, a standing wave is formed by the combination of the sound waves traveling in opposite directions. According to the physics of sound, the open ended cylinder resonates as air passes through it from the Marimba bar if and only if its acoustical length is approximately equal to half of the sound wavelength.

A Marimba resonator often imposes its resonance frequency on the system, hence the resonators are sized for each specific Marimba bar note where the frequencies produced by the bars and the resonators are added constructively, thereby causing the sound produced to resonate to its maximum amplitude (Rienstra and Hirschberg, 2004).

If a cylindrical air column is open at both ends, antinodes of air displacement (the nodes of air pressure) must appear at the ends of the tube for a standing wave to occur because air at both ends of the column is relatively free to move. For air in an open pipe to vibrate in a standing wave pattern, the two required antinodes have to be at the ends of the pipe and a node in the middle of the pipe. This leads to the tube length being a multiple of half-wavelengths (i.e., the distance between antinodes will always be $\frac{\lambda}{2}$, λ or $\frac{3\lambda}{2}$ etc).

Tracey (1969) points out that, Marimba resonators are designed so as to resonate with the fundamental frequency. Thus, a resonator's length is determined by the note at which it must resonate. A resonator tube is tuned to a certain note with a fundamental frequency by adjusting its length.

When the open upper end of a cylinder in a Marimba is excited by the harmonic vibration from a bar placed immediately above it, the bar vibrates and causes the air to vibrate. Sound is caused by tiny variation in air pressure as it vibrates above and below the ambient atmospheric pressure that can be detected by some receiver. Normally sound pressure amplitudes range from a few micro-pascals to a few hundred Pascal that fluctuate around atmospheric pressure $\left(1.01\times10^{5}\ pa\right)$.

This variation in air pressure propagates in the form of longitudinal waves throughout the medium (air) at a velocity called the sound speed. The variations of pressure oscillate very rapidly in time around the atmospheric pressure and take place in the audio-range that lies between 20 and 20,000 cycles per second as perceived by the human ear. These tiny variations in air pressure result in a series of high-pressure and low-pressure regions (Figure 1.2). Since the source of the sound waves vibrates sinusoidally, the pressure variations are also sinusoidal.

Coulson (1977) points out that, when two waves of equal wavelength and amplitude propagate in opposite directions as a result of interference, the two waves are superimposed on each other and the resulting wave is called a standing wave. In a standing wave the particles of the medium at certain points do not oscillate. These points are called nodes. At certain points, called antinodes the particles of the medium have maximum amplitude of oscillation.

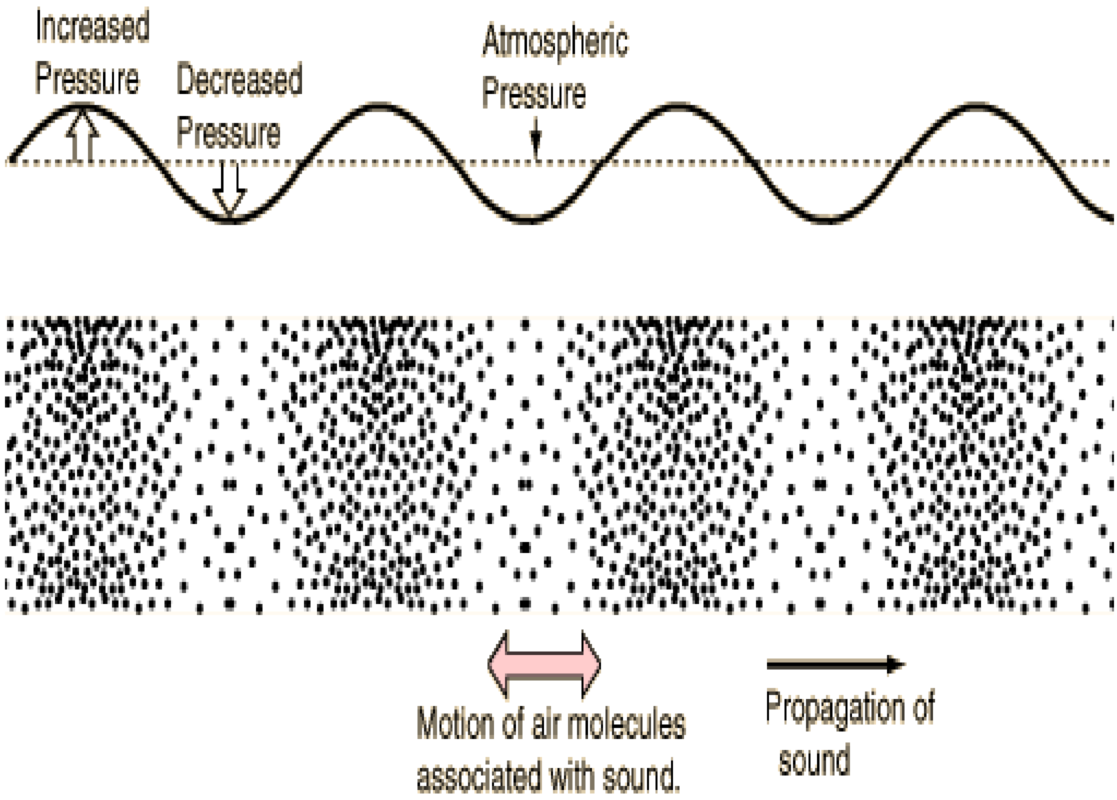

Figure 1.2: Pressure Variation in a Tube

The study of sound wave amplitude, frequency, wavelength, and velocity can be explained by considering the sketch in Figure 1.3.

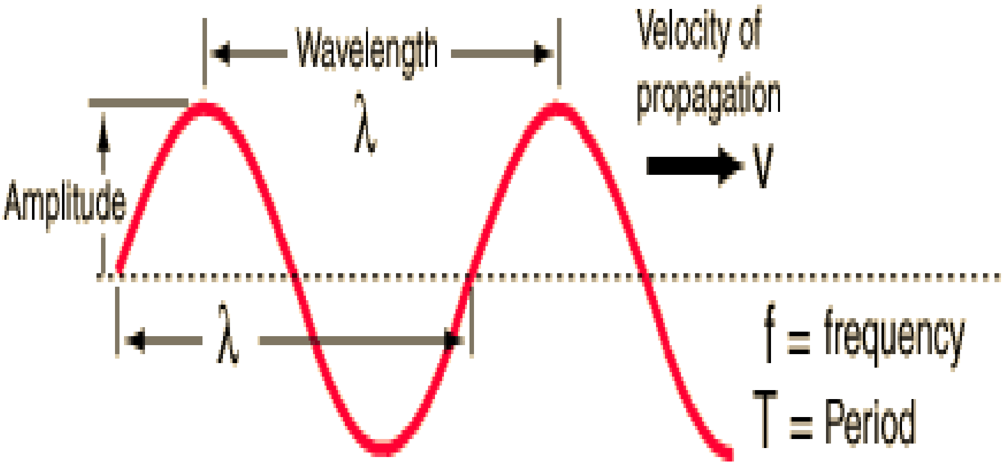

Figure 1.3: Propagation of Sound Waves in Air

Sound waves that consist only of a pure tone are characterized by the amplitude of pressure changes, which can be described by the maximum pressure or the root-mean-square amplitude, and is expressed in Pascal (Pa). Root-mean-square means the instantaneous sound pressures (which can be positive or negative) are squared averaged and the square root of the average is taken.

(i) The wavelength λ, is the distance travelled by the wave during one cycle.

(ii) The frequency f, is the number of cycles per unit time and is expressed in Hertz (Hz).

The sound produced by a Marimba key is soft, depending on the type of wood used. The manufacturer adds something that will amplify the sound called a resonator (Vanessa, 2005). Musically, the Marimba resonator is important because it directly influences the overall sound quality of a Marimba system. The vibrating Marimba bars do not radiate sound very efficiently; hence in order to amplify the sound, the resonators are mounted underneath the bars (Fletcher and Rossing, 1991).

This study aims at analyzing the influence of the natural frequencies of a cylinder open at both ends, specifically Marimba resonators, on the overall sound quality of a Marimba. This is done by developing a mathematical model of resonance in a cylinder that is open at both ends.

The mathematical model for analyzing the natural frequencies of the open ended cylinder as applied to Marimba resonator is derived using the theory of linear acoustic wave equation, which is reduced to Helmholtz equation and solved numerically using the finite difference method (FDM).

Due to the complexity of obtaining an analytical solution and the nature of the problem domain, a numerical method is used to solve the mathematical model. The finite difference method is chosen to approximate the solution of the domain function because it is simple to implement over a domain with regular geometry.

4.4 Statement of the Problem

The resonator of a Marimba is a three dimensional circular cylinder with uniform cross-section area tuned to the fundamental mode of its corresponding bar. Fletcher and Rossing (1991) points out that, the purpose of the resonator is to amplify the sound of the entire Marimba system.

Few studies have been carried on mathematical modelling of resonance in an open-ended cylinder. This study aims at analyzing the sensitivity of natural frequencies to small variations in the dimensions (length, radius) of a cylinder open at both ends, as applied to the Marimba resonator. This is achieved by developing a mathematical model of resonance in a cylinder that is open at both ends using the finite difference method as applied to the resonator of a Marimba.

1.2 Research Objectives

The general objective of this study was to develop and analyses a mathematical model of resonance in a cylinder open at both ends using the finite difference method, and apply it to the Marimba resonator. This was achieved by targeting the following specific objectives:

(i) To develop a mathematical model for the movement of air in a cylinder open at both ends with application to the Marimba resonator.

(ii) To determine the sensitivity of the natural frequencies to small variations in the dimensions (length, radius) of the cylindrical resonator.

(iii) To validate the modal developed.

1.3 Significance of the Study

The following are the aspects that make this study significant:

(i) **Contribution to Knowledge**

This study will contribute a mathematical model for analyzing the movement of air in a Marimba resonator.

(ii) **Contribution to Policy Makers**

The resulting established relationship between resonant (natural) frequencies of cylinders and dimensions of the cylinders is useful to Marimba and idiophone manufacturers in promoting the use of such musical instrument.

(iii) **Contribution to Practitioners**

Stakeholders interested in studying resonance in open ended cylinders of different dimensions may use the finding in this dissertation as a basis for further studies

1.4 Scope and Limitation of the Study

This study focuses only on resonance in an open ended cylinder, caused by sinusoidal vibrations of air of a struck Marimba bar as a source of sound. The variation in air pressure in the cylinder propagates as longitudinal waves through the medium.

1.5 Organization of the Dissertation

This dissertation has five main chapters: Introduction; Literature Review; Methodology (formulation of a mathematical model of resonance in open-ended cylinder); Numerical simulation of the model; Discussion, Conclusion, Recommendations and future work.

The first chapter gives an overview of the Marimba resonator together with the issues associated with resonance in open-ended cylinders in relation to the theory and laws that govern standing waves. The second chapter reviews the literature on the subject and highlight on already established facts regarding resonance in open ended cylinders as related to this research. A lot of efforts are made to match the literature with the current problem (i.e, the effects of small variations in the dimensions (length, radius) of an open ended cylinder to its natural frequencies).

Chapter three covers the method used in this study; it describes the mathematical model of resonance in an open-ended cylinder.

Chapter four focuses on numerical simulation of the model. The numerical and laboratory experimental work is explained, including, determination of the relationship between the natural frequency and the dimensions of an open ended cylinder.

The last chapter summarizes the findings of this study. It reveals that, the small variations of the length of an open-ended cylinder have significant effect on the natural frequency of the Marimba resonator and hence on its musical quality. Moreover, it was also established that, small variations of the radius of the cylinder does not seriously affect its natural frequency unless it becomes very large.

CHAPTER TWO

LITERATURE REVIEW

The natural sound produced by a marimba key is soft, depending on the type of wood that has been used, so the maker adds something that will make that sound louder called a resonator. Some African cultures use gourds or animal horns or even banana stems to amplify the sound (Vanessa, 2005).

Fletcher and Rossing (1998) emphasized that, vibrating bars do not radiate sound very efficiently which causes Marimba builders to mount resonators underneath the bars of marimba in order to amplify the radiation of sound from bars.

The most common Marimba resonator is a straight cylindrical tube made of bamboo, metal, or Poly Vinyl Chloride (PVC)-cylinders (Forster, 2010). The straight cylindrical resonator is used in steady of the cavity resonator because its mathematics is not far more complicated and that, the length and frequency equations for cylindrical resonators are easy to understand. When the Marimbas bar is vibrating just below the open end of a cylinder, sound waves are sent into the column of air inside the cylinder whereby they reflect off at the closed end of the cylinder and travel back to the opening (Fletcher and Rossing, 1991).

Nederveen (1998) show that, reflected waves combine with new waves coming from the bar which is being struck. Resonance occurs if the reflected waves and the new waves are in phase with each other (constructive interference) to create a standing wave (Boelkes and Hoffmann, 2011). The amplitudes of the reflected and new waves combine and a louder sound with a definite frequency (note) is heard by which serve the work of combining the tones with the struck bars and amplify them. The column of air vibrates at its natural or resonant frequency to produce the sound.

Sound waves are longitudinal waves where the speed of sound in air varies from about 330m/s to 340 m/s depending upon the density, pressure and temperature of the air. The theoretical value for the speed of sound in air can be found using the equation:

$$c = (331\,\mathrm{m/s})\sqrt{1 + \frac{T_c}{273°C}} \quad . \tag{2.1}$$

Where T_c is the air temperature in °C and the speed of sound (c) will be given in m/s

The standing wave in an open tube has displacement antinodes and pressure nodes at the open ends of the tube (Fletcher and Rossing, 1998). The condition for resonance and frequencies at which standing waves is supported in a tube of length L is given by

$$L = n\frac{\lambda}{2}\ , \quad n=1,2,3,... \tag{2.2}$$

So that

$$f_n = n\left(\frac{c}{2L}\right), \quad n=1,2,3,... \tag{2.3}$$

Beranek (1988) point out that, the wave equation which describes the behavior of sound waves propagating in any acoustic medium results from the combination of several laws of physics, namely Newton's second law of motion, the gas law and the laws of conservation of mass.

(Fletcher and Rossing, 1991) point out that, the wave equation can be derived from two fundamental laws of the theory of continuum mechanics. These two are the principle of conservation of mass and the principle of balance of momentum. Moreover, Beranek (1988) emphasised that, the wave equation can be derived through the combinations of established equations expressing Newton's law of motion (the Euler`s equation as the balance of momentum), the gas laws and the laws of conservation of mass. Sound waves in a fluid (air) are oscillatory disturbances, generated, for example, by a vibrating surface. Such disturbances manifest themselves by pressure fluctuations, which is what we can hear, but also by similar perturbations of the temperature and the density, and by particles moving back and forth.

Mendonca (2007) in his thesis states that, if the acoustic pressure is assumed to be harmonic then the wave equation simplifies to the Helmholtz equation which serves as the starting point for the finite difference model formulation.

The time-dependent wave equation governing the acoustic field can be reduced to the Helmholtz equation when harmonic solutions are considered (Beranek and Vér, 2005). Moreover, they emphasized that, the acoustic field is assumed to be present in the domain of a homogeneous isotropic fluid whatever the shape and nature of the domain, the acoustic field is taken to be governed by the linear wave equation where the differential equation for pressure field associated with acoustic vibrations in two dimensional rooms with rigid boundaries is given by

$$\nabla^2 u + \frac{w^2}{c^2} u = 0 \tag{2.4}$$

Equation (2.4) is the time-independent partial differential equation called Helmholtz

Marderness (2012) points out that, the Helmholtz equation (2.4) is a special case of the wave equation which is only a function of position, as opposed to a function of position and time. This equation describes standing waves within a fluid volume.

CHAPTER THREE

FORMULATION OF A MATHEMATICAL MODEL OF RESONANCE IN AN OPEN-ENDED CYLINDERS

3.1 Properties of Sound Waves

A sound wave is characterized by its frequency f measured in Hertz ; wavelength λ measured in meters; period T measured in seconds ; sound propagating speed c measured in metres per second; and by its amplitude, which is a nonnegative scalar measure of the wave's maximum oscillation (or maximum disturbance) in the medium during a single wave oscillation.

The period T is the time taken for one oscillation of a wave to pass a fixed point. It is related to frequency by:

$$T = \frac{1}{f} \; . \qquad (3.1)$$

The speed of sound propagation, the frequency and the wavelength are related through the equation

$$c = f\lambda \, . \qquad (3.2)$$

In general, the quantities frequency f , wavelength λ , the speed of sound c and period T are related through the equation:

$$f = \frac{1}{T} = \frac{c}{\lambda} = kc \Rightarrow k = \frac{\omega}{c} = \frac{2\pi}{\lambda} \, . \qquad (3.3)$$

where k and ω denote the wave number and angular frequency, respectively .

3.2 Derivation of the Equations Governing Resonance in an Open Ended Cylinder

In physics, resonance is the tendency of a system to oscillate with greater amplitude at some frequencies than at others. Frequencies at which the response amplitude is a relative maximum are known as the system's resonant frequencies. The physical laws of acoustic standing waves in a cylinder are used to derive a general formula for calculating resonant frequencies of a cylinder that is opened at both ends.

Consider a cylinder of length L open at both ends. A standing wave in such a cylinder must have pressure nodes (displacement antinodes) at both ends. If there are no additional pressure nodes in the middle of the cylinder, the distance between the two nodes at the cylinder ends is half the wavelength (Fletcher and Rossing, 1991). This leads to the equation

$$L = \frac{\lambda}{2}. \tag{3.4}$$

This equation represents the wavelength λ_1 at the fundamental harmonic defined by the equation:

$$\lambda_1 = 2L. \tag{3.5}$$

so that, a cylinder with n harmonics will have a wavelength

$$\lambda_n = \frac{2L}{n}, \quad n=1,2,3,... \tag{3.6}$$

The simplest analytic formula, which determines the eigenfrequencies of an air column resonating in an open ended cylinder is given by the equation;

$$f_n = n\left(\frac{c}{2L}\right), n = 1,2,3, \tag{3.7}$$

where f_n is the frequency of the $n-th$ eigenmode, c is speed of sound propagation in (m/s) and L denotes a length of cylinder measured in metres.

when n=1 the equation (3.7) becomes a fundamental frequency f_1 given by;

$$f_1 = \frac{c}{2L} \quad (3.8)$$

Equation (3.7) can be re-arranged to give the physical length of a cylinder needed to resonate at a certain frequency

$$L = n\left(\frac{c}{2f_n}\right), n = 1, 2, 3, \ldots \quad (3.9)$$

Marimba resonators are designed to resonate with the fundamental frequency f_1 corresponding to n equal to 1, in which case

$$L = \frac{c}{2f_1}. \quad (3.10)$$

Thus, a resonator's length is determined by the note it is intended to resonate; and a resonator cylinder is tuned to a certain note with a fundamental frequency f_1 by adjusting its length L as per equation (3.10).

3.3 The Function Used to Model Resonance in an Open-Ended Cylinder

The three functions which can be used to model resonance in open ended cylinder are;

(i) Particle velocity.

(ii) Velocity potential.

(iii) Acoustic pressure.

3.3.1 Acoustic Pressure

This is the pressure caused by the vibrations of air when the Marimba bar is struck. It is given by the equation;

$$p\left(x,y,z,t\right)=i\omega\rho u(\mathrm{x},\mathrm{y},\mathrm{z})e^{-i\omega t} \qquad (3.11)$$

where p , ω , ρ and u are pressure, angular frequency, density of air and velocity potential respectively.

3.3.2 Particle Velocity

The particle velocity $\tilde{V}\left(x,y,z,t\right)$ is the physical speed of a parcel of air as it oscillates in the direction where the sound wave is propagating. The particles of the medium oscillate around their original position with a relatively small particle velocity. The Particle velocity is given through the equation;

$$\tilde{V}\left(x,y,z,t\right)=\tilde{v}\left(x,y,z\right)e^{-i\omega t} \quad . \qquad (3.12)$$

where $\tilde{v}$ denotes particle velocity amplitude

3.3.3 Velocity potential

The velocity potential $\psi\left(x,y,z,t\right)$ is a scalar function whose gradient is equal to the particle velocity of the fluid (air) at that point. Velocity potential is given by the equation;

$$\psi\left(x,y,z,t\right)=u\left(x,y,z\right)e^{-i\omega t} \quad . \qquad (3.13)$$

where $u\left(x,y,z\right)$ is the velocity potential amplitude due to pressure variations.

3.4 Relationship between Velocity Potential and Particle Velocity

If a fluid is frictionless (non-viscosity), incompressible and adiabatic, its particle velocity as a function of position is described by a velocity potential given by the equation:

$$\nabla u\left(x,y,z\right)=\tilde{v}\left(x,y,z\right). \tag{3.14}$$

3.5 Theoretical Basis

3.5.1 The Wave Equation

Kirkup (2007) points out that, whatever the shape and nature of the problem domain, the acoustic field is taken to be governed by the standard linear wave equation which in terms of the velocity potential is given by

$$\frac{\partial^2}{\partial t^2}\psi\left(x,y,z,\mathrm{t}\right)=c^2\nabla^2\psi\left(x,y,z,\mathrm{t}\right). \tag{3.15}$$

Equation (3.12) can be reduced to the well known Helmholtz equation, which greatly simplifies the complexity of the problem and serves as the starting point for the finite difference model formulation (Kirkup, 2007).

3.5.2 The Helmholtz Equation

The model is based on the theory of the Helmholtz equation. The Helmholtz equation is a special case of the wave equation which is used to describe stationary waves within a fluid volume.

3.6 Derivation of Helmholtz Equation from the Wave Equation

The Helmholtz equation is a special case of the wave equation which is only a function of position, as opposed to a function of position and time. The Helmholtz

Equation describes stationary waves within a fluid volume. The Helmholtz equation is obtained from the wave equation (3.15) by assuming a solution of the form in (3.16)

$$\psi\left(x,y,z,t\right)=\mathrm{Re}\left(e^{-i\omega t}u\left(x,y,z\right)\right),\, i^{2}=-1\,. \qquad (3.16)$$

where $u\left(x,y,z\right)$ denotes some complex valued scalar function while Re means that the real part is taken, because $\psi\left(x,y,z,t\right)$ is a complex valued function.

Differentiating Equation (3.16) with respect to time twice yields

$$\frac{\partial^{2}\psi\left(x,y,z,t\right)}{\partial t^{2}}=\left(-i\omega\right)\left(-i\omega\right)u\left(x,y,z\right)e^{-i\omega t}=-\omega^{2}u\left(x,y,z\right)e^{-i\omega t}\,. \qquad (3.17)$$

Substituting Equations (3.17) and (3.16) into Equation (3.15) and re-arranging it yields;

$$\nabla^{2}ue^{-i\omega t}+\frac{\omega^{2}}{c^{2}}ue^{-i\omega t}=0\,. \qquad (3.18)$$

If the exponential time variable $e^{-i\omega t}$ is cancelled, and using the definition of wave number (3.3) gives the Helmholtz equation

$$\nabla^{2}u\left(x,y,z\right)+k^{2}u\left(x,y,z\right)=0\,. \qquad (3.19)$$

3.7 Helmholtz's Equation in Cylindrical Coordinates

Marimba resonators are cylindrical in their design. It is therefore convenient to express the underlying Helmholtz equation (3.19) in cylindrical coordinates. Cylindrical coordinates (r,θ,z) are related to the Cartesian coordinates (x,y,z) by the equations

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$.

Transformation of the Helmholtz equation $\nabla^2 u(x,y,z) + ku(x,y,z) = 0$ essentially implies transforming the equation $\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 z}{\partial z^2}$ into a corresponding expression involving cylindrical coordinates. This is achieved by finding differential operators involving the cylindrical coordinates that correspond to the cartesian differential operators are $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

By applying the chain rule for differentiation one finds

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta}\dfrac{\partial \theta}{\partial y} \end{cases}. \tag{3.20}$$

In order to obtain expressions for the terms $\frac{\partial r}{\partial x}, \ \frac{\partial \theta}{\partial x}$ and $\frac{\partial r}{\partial y}, \ \frac{\partial \theta}{\partial y}$ that appear in equation (3.19) one first differentiates partially the transformation equations $x = r\cos\theta$ and $y = r\sin\theta$ to get

$$\begin{cases} \dfrac{\partial}{\partial x}(x) = \dfrac{\partial}{\partial x}(r\cos\theta) = \cos\theta\dfrac{\partial r}{\partial x} - r\sin\theta\dfrac{\partial \theta}{\partial x} = 1 \\ \dfrac{\partial}{\partial x}(y) = \dfrac{\partial}{\partial x}(r\sin\theta) = \sin\theta\dfrac{\partial r}{\partial x} + r\cos\theta\dfrac{\partial \theta}{\partial x} = 0 \end{cases}. \tag{3.21}$$

and

$$\begin{cases} \dfrac{\partial}{\partial y}(x) = \dfrac{\partial}{\partial y}(r\cos\theta) = \cos\theta\dfrac{\partial r}{\partial y} - r\sin\theta\dfrac{\partial \theta}{\partial y} = 0 \\ \dfrac{\partial}{\partial y}(y) = \dfrac{\partial}{\partial y}(r\sin\theta) = \sin\theta\dfrac{\partial r}{\partial y} + r\cos\theta\dfrac{\partial \theta}{\partial y} = 1 \end{cases}. \tag{3.22}$$

Equations (3.20) and (3.21) are a pair of simultaneous linear equations

$$\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}r_x\\ \theta_x\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix},\quad \text{and}\quad \begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}r_y\\ \theta_y\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

whose solutions are readily found to be

$$\frac{\partial r}{\partial x}=\cos\theta,\quad \frac{\partial\theta}{\partial x}=-\frac{\sin\theta}{r},\quad \frac{\partial r}{\partial y}=\sin\theta,\quad \frac{\partial\theta}{\partial y}=\frac{\cos\theta}{r}. \tag{3.23}$$

Substituting these results into equation (3.19) leads to

$$\begin{cases}\dfrac{\partial u}{\partial x}=\cos\theta\dfrac{\partial u}{\partial r}-\dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial\theta}\\[2ex] \dfrac{\partial u}{\partial y}=\sin\theta\dfrac{\partial u}{\partial r}+\dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial\theta}\end{cases}. \tag{3.24}$$

Equation (3.23) is a pair of equations, each expressing one of the two first order partial derivatives of u with respect to the cartesian coordinates x, y in terms of first order partial derivatives of u with respect to the cylindrical coordinates r, θ. To express $\dfrac{\partial^2 u}{\partial x^2}, \ \dfrac{\partial^2 u}{\partial y^2}$ in cylindrical coordinates one starts by noting that

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right). \tag{3.25}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right). \tag{3.26}$$

The differential operators $\dfrac{\partial}{\partial x}, \ \dfrac{\partial}{\partial y}$ which appear in these equations are obtained by removing the function u from each of the pair of equations in (3.24). This gives

$$\begin{cases} \dfrac{\partial}{\partial x} = \cos\theta \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial\theta} \\ \dfrac{\partial}{\partial y} = \sin\theta \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial\theta} \end{cases} .$$

Substituting these operators into (3.24) and (3.25) leads to the results

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= c^2\frac{\partial^2 u}{\partial r^2} + \frac{sc}{r^2}\frac{\partial u}{\partial\theta} - \frac{sc}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{s^2}{r}\frac{\partial u}{\partial r} - \frac{sc}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{sc}{r^2}\frac{\partial u}{\partial\theta} + \frac{s^2}{r^2}\frac{\partial^2 u}{\partial\theta^2} . \tag{3.27}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= s^2\frac{\partial^2 u}{\partial r^2} - \frac{sc}{r^2}\frac{\partial u}{\partial\theta} + \frac{sc}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{c^2}{r}\frac{\partial u}{\partial r} + \frac{sc}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{sc}{r^2}\frac{\partial u}{\partial\theta} + \frac{c^2}{r^2}\frac{\partial^2 u}{\partial\theta^2} . \tag{3.28}$$

Where for simplicity's sake the abbreviations $s = \sin\theta$ and $c = \cos\theta$ have been used. Finally, adding equations (3.26) and (3.27), and from the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, one gets $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}$.

With this result, the equivalence of Helmholtz's equation

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} + ku(x,y,z) = 0 .$$

in cylindrical coordinates is

$$\frac{\partial^2 u(r,\theta,z)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta,z)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta,z)}{\partial\theta^2} + \frac{\partial^2 u(r,\theta,z)}{\partial z^2} + ku(r,\theta,z) = 0 . \tag{3.29}$$

In the case of the cylindrical Marimba resonator the function u depends only on the distance r from the axis of symmetry and on the value of z but is independent of the angle θ. This implies that the term $\frac{\partial^2 u}{\partial \theta^2}$ is identically zero and hence equation (3.29) reduces to

$$\frac{\partial^2 u(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,z)}{\partial r} + \frac{\partial^2 u(r,z)}{\partial z^2} + k^2 u(r,z) = 0\,. \qquad (3.30)$$

Fletcher et al., (2005) show that, since it is assumed that there is no angular variation around the pipe, the coordinate θ can be dropped and therefore when the Helmholtz equation (3.19) is expressed in cylindrical coordinates (3.31), then $u(r,\theta,z)$ reduced to a two dimensional functional $u(r,z)$, which is vector of the two variables r and z.

$$\frac{\partial^2 u(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,z)}{\partial r} + \frac{\partial^2 u(r,z)}{\partial z^2} + k^2 u(r,z) = 0\,, 0 < r < R, 0 < z < Z\,. \qquad (3.31)$$

3.8 The Numerical Model

For an open ended cylinder, the acoustic modal characteristics (natural or resonant frequencies, eigenmode shapes, etc.) are determined by solving numerically the Helmholtz equation (3.31) subject to appropriate boundary conditions. Since the open end boundaries are parallel, the modal characteristics in the axial direction can be found, and the problem domain can be viewed as a rectangular region $0 \le z \le Z, 0 \le r \le R$. Thus, a numerical method based either on finite differences or the finite element may be used to solve this boundary value problem. In this dissertation, the finite difference method is used because it is relatively straight forward to derive, simple to implement and is sufficiently accurate on a rectangular domain compared to the finite element method. The finite difference method is a numerical technique which converts the underlying differential equation to an approximate finite difference equation by replacing partial derivatives with corresponding finite difference quotients.

3.8.1 Discretization of the Helmholtz Equation Using the FDM

Inside the open-ended cylinder (Figure 3.2), the volume is bounded by the two opposite parallel cross- section surfaces. For the resulting cross-sectional problem, half of the cylinder cross-section will be considered since it is symmetrical about the axis of the cylinder. The problem is then reduced to solving the two dimensional Helmholtz equation (3.34) over a rectangular plane surface 0ABC (Figure 3.1).

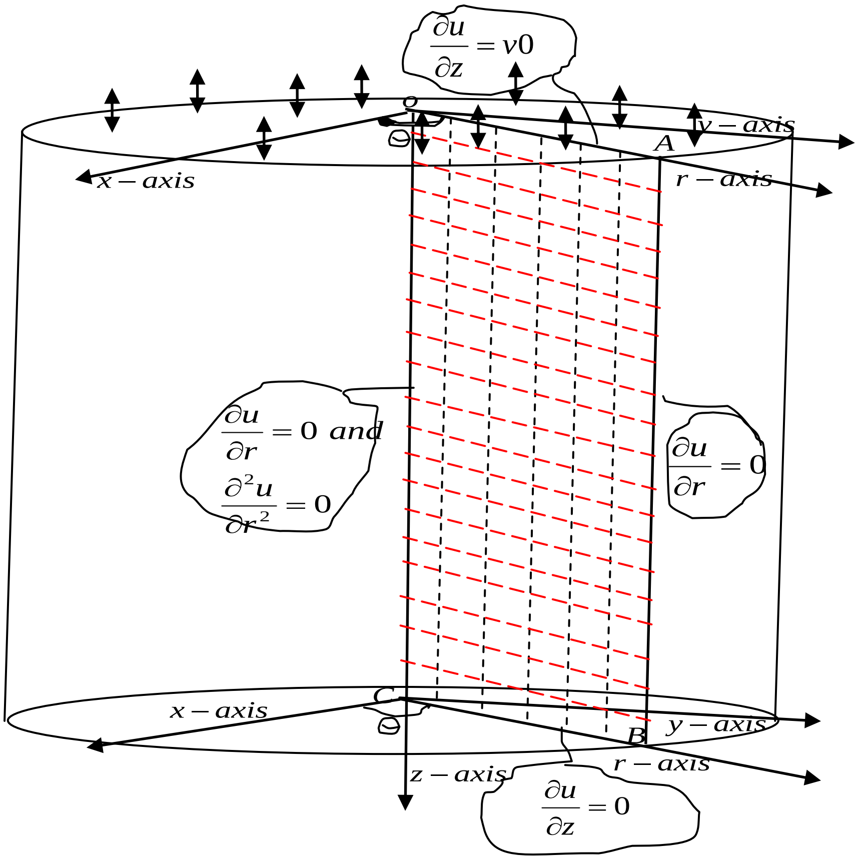

Figure 3. 1: The Problem Domain

The illustration of the problem domain above Figure 3.2 can be simplified to Figure 3.3 where the three point central difference formula was applied.

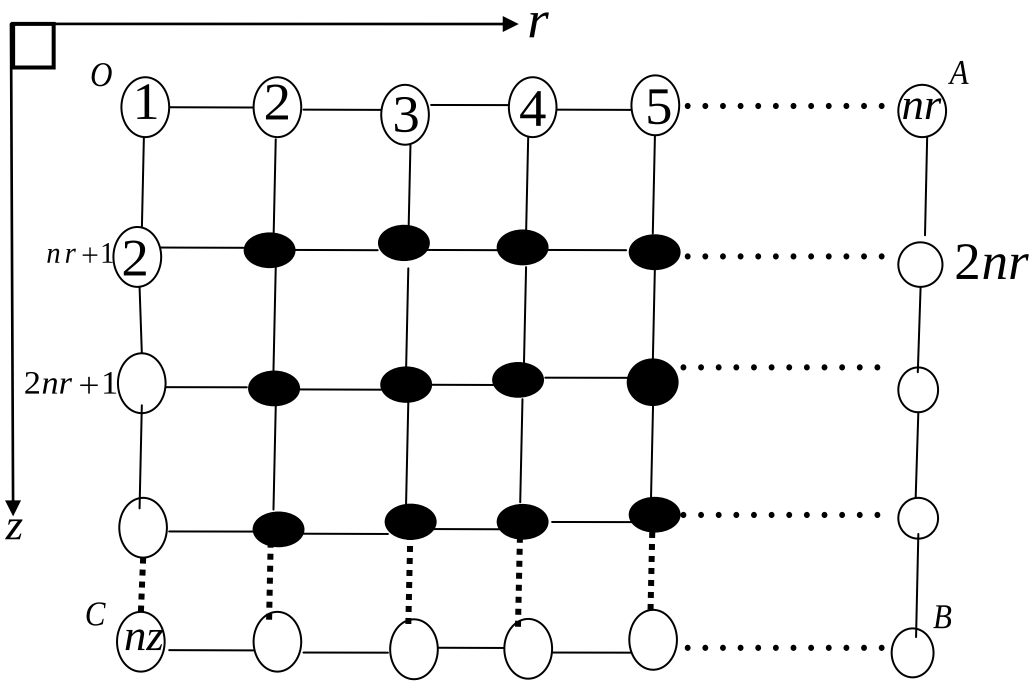

Figure 3. 2: Computational Grid OABC of rz-Rectangular Plane

When we apply the three point central difference formula to Figure 3.3, yields the finite difference equations (3.35) up to (3.38).

Note: $u_{i,j} = u(r_i, z_j)$ is a pressure of air at the coordinates (r_i, z_j) so that the Finite difference equations becomes;

$$\frac{\partial u}{\partial z}(r_i, z_j) \cong \left(\frac{u_{i,j+1} - u_{i,j-1}}{2h_z}\right) + O\left(h_z^2\right). \qquad (3.32)$$

$$\frac{\partial u}{\partial r}(\mathrm{r}_i, \mathrm{z}_j) \cong \left(\frac{u_{i+1,j} - u_{i-1,j}}{2h_r} \right) + O\left(h_r^2\right). \tag{3.33}$$

$$\frac{\partial^2 u}{\partial r^2}\left(r_i, z_j\right) = \left(\frac{u_{i-1,j} - 2\mathrm{u}_{i,j} + u_{i+1,j}}{h_r^2} \right) + O\left(h_r^2\right). \tag{3.34}$$

$$\frac{\partial^2 u}{\partial z^2}\left(r_i, z_j\right) = \left(\frac{u_{i,j+1} - 2\mathrm{u}_{i,j} + u_{i,j-1}}{h_Z^2} \right) + O\left(h_z^2\right). \tag{3.35}$$

Each term in the Helmholtz equation (3.31) is approximated by an appropriate finite difference quotient to lead to a corresponding finite difference equation. At any point (r_i, z_j) where $i = 1,2,3,4 \ldots \ldots .. n_{r,}$ and $j = 1,2,3,4 \ldots \ldots .. n_z$ in the discretized problem domain (Figure 3.2) with nr =16 nodes and nz =20 nodes

we have

$$\frac{\partial^2 u(r_i, z_j)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r_i, z_j)}{\partial r} + \frac{\partial^2 u(r_i, z_j)}{\partial z^2} + k^2 u(r_i, z_j) = 0\,. \tag{3.36}$$

We use second order central difference quotients to approximate both first and second order partial derivatives and get

$$\left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_r^2} \right) + \frac{1}{r_i}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2h_r} \right) + \left(\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_z^2} \right) + k^2 u_{i,j} = 0\,. \tag{3.37}$$

Equation (3.47) is simplified to give a finite difference model for the Helmholtz equation ;

$$\left(\frac{1}{h_r^2} - \frac{1}{2h_r r_i} \right) u_{i-1,j} + \left(k^2 - \frac{2}{h_r^2} - \frac{2}{h_z^2} \right) u_{i,j} + \left(\frac{1}{h_r^2} + \frac{1}{2h_r r_i} \right) u_{i+1,j} + \left(\frac{1}{h_z^2} \right) u_{i,j-1} + \left(\frac{1}{h_z^2} \right) u_{i,j+1} = 0\,. \tag{3.38}$$

If we let $a_i = \frac{1}{h_r^2} - \frac{1}{2h_r r_i}$, $g = k^2 - \frac{2}{h_r^2} - \frac{2}{h_z^2}$, $c_i = \frac{1}{h_r^2} + \frac{1}{2h_r r_i}$ and $b = d = \frac{1}{h_z^2}$ then

equation (3.38) becomes

$$a_i u_{i-1,j} + g u_{i,j} + c_i u_{i+1,j} + b u_{i,j-1} + d u_{i,j+1} = 0. \qquad (3.39)$$

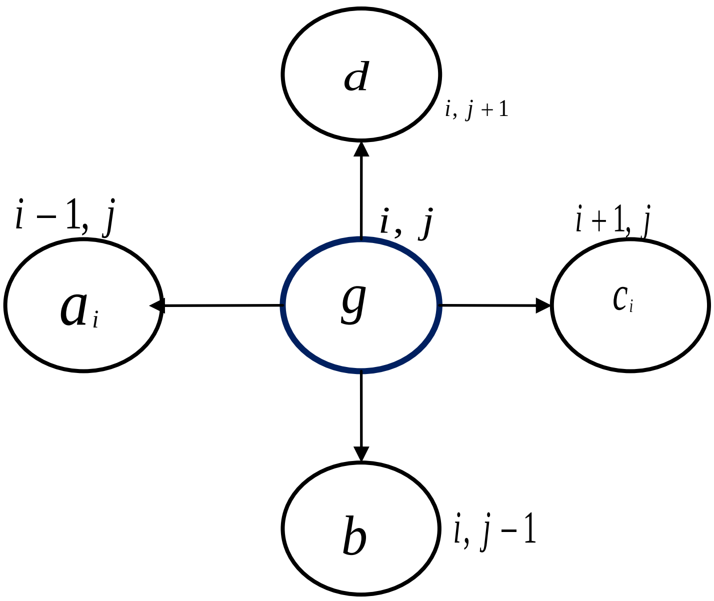

Figure 3. 3: Computational Molecule at Interior Nodes

3.8.2 Boundary Condition along the Axis of the Cylinder

The term $\frac{1}{r}\frac{\partial u}{\partial r}$ in the Helmholtz equation (3.31) is discontinuous at $r = 0$. However,

because $\frac{\partial u}{\partial r} = 0$ at $r = 0$ (line of symmetry), we can apply L´Hopital`s rule

$$\lim_{r\to 0}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) = \lim_{r\to 0}\left(\frac{\partial^2 u}{\partial r^2}\right) = u''(0,z) = u''(0). \qquad (3.40)$$

Therefore at $r=0$ the term becomes $\frac{\partial^2 u}{\partial r^2}=0$ thus, the boundary conditions which models the axis of the cylinder are $\frac{\partial^2 u}{\partial r^2}=0$ and $\frac{\partial u}{\partial r}=0$, which in terms of finite differences becomes

$$u_{i-1,j}-2u_{i,j}+u_{i+1,j}=0\,. \tag{3.41}$$

and $$u_{i+1,j}-u_{i-1,j}=0\,. \tag{3.42}$$

Equations (3.41, 3.42 and 3.39) together imply

$$gu_{i,j}+(a_i+c_i)u_{i+1,j}+bu_{i,j-1}+du_{i,j+1}=0\,. \tag{3.43}$$

For any point (r_i,z_j) on the axis the equation (3.43) is used to model the symmetry axis of the cylinder.

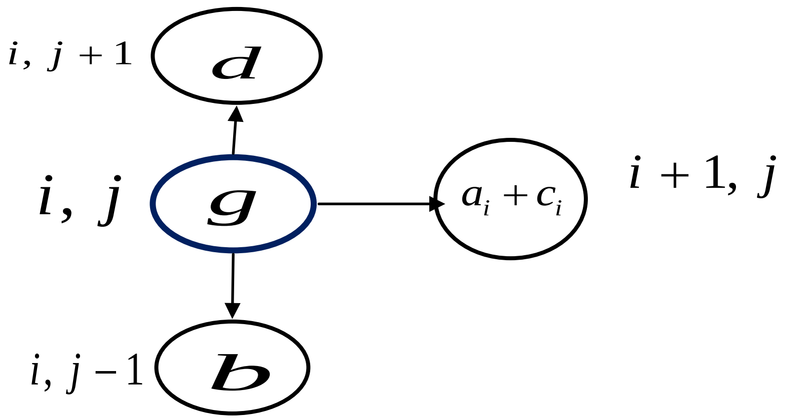

Figure 3. 4: Computational Molecule along the Symmetry Axis

3.8.3 Boundary Condition at the Cylinder Wall

Consider the relationship between particle velocity and velocity potential from equation (3.14)

$$\nabla u = \tilde{v} \Rightarrow \nabla u \cdot \tilde{n} = \tilde{v} \cdot \tilde{n} \Leftrightarrow \frac{\partial u}{\partial n} = 0 . \qquad (3.44)$$

where $\tilde{v}$ is the particle velocity and $\tilde{n}$ is the unit vector normal to the curved surface of the cylinder. The normal derivative $\frac{\partial u}{\partial n}$ is equal zero because the particle velocity has no component in the normal direction to the fixed surface of the cylinder.

$$\frac{\partial u}{\partial n} = 0 \Rightarrow \frac{\partial u}{\partial r} = 0, \text{ so that } \frac{\partial u}{\partial r} \cong \left(\frac{u_{i+1,j} - u_{i-1,j}}{2h_r} \right) = 0, \Leftrightarrow u_{i+1,j} - u_{i-1,j} = 0 . \qquad (3.45)$$

Equation (3.45) implies $u_{i+1,j} = u_{i-1,j}$

Substituting this result into the general equation (3.39) we get

$$\left(a_i + c_j \right) u_{i-1,j} + g u_{i,j} + b u_{i,j-1} + d u_{i,j+1} = 0 . \qquad (3.46)$$

Therefore, the equation (3.46) is used to models the particle velocity at the boundary of the cylinder.

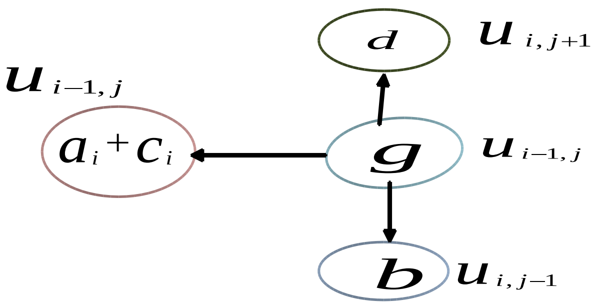

Figure 3. 5: Computational Molecule along the Hard Wall Cylinder Boundary

3.8.4 Boundary Condition on the Open Top Surface of the Cylinder

The boundary condition on the open top surface of the cylinder just below the Marimba bar is that, the air vibrates with the constant velocity v_0 of the struck wooden bar with a frequency of middle C. Considering the relationship between velocity potential u and particle velocity $\tilde{v}$ from equation(3.14) , the boundary condition is modeled by :-

$$\nabla u = \tilde{v}, \Rightarrow \nabla u \cdot \tilde{n} = \tilde{v} \cdot \tilde{n} \Leftrightarrow \frac{\partial u}{\partial n} = v_0, \Rightarrow \frac{\partial u}{\partial z} = v_0 . \tag{3.48}$$

Where $\tilde{n}$ denotes a unit normal vector and v_0 is the magnitude of air velocity

In terms of finite difference equation, we get

$$\frac{\partial u}{\partial z} \cong \left(\frac{u_{i,j+1} - u_{i,j-1}}{2h_z} \right) = v_0 \Rightarrow u_{i,j+1} = u_{i,j-1} + 2h_z v_0 . \tag{3.49}$$

Substitute this into equation (3.39) leads to

$$a_i u_{i-1,j} + g u_{i,j} + c_i u_{i+1,j} + (b+d) u_{i,j-1} = 2h_z v_0 . \tag{3.50}$$

The equation (3.50) is used to model the solution across the top surface of the cylinder.

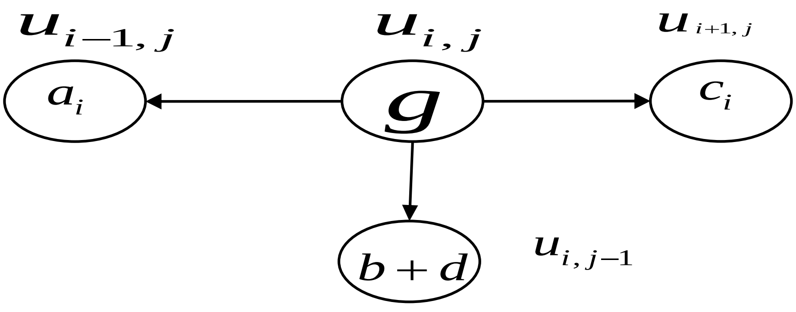

Figure 3.6: Computational Molecule along the Top Cylinder Boundary

3.8.5 Boundary Condition on the Open Bottom Surface of the Cylinder

The boundary condition on the open bottom surface of the cylinder is obtained by noting that, when sound reaches the bottom from the top of the cylinder, the pressure is just the ambient pressure and the sound pressure is gradually dying down to zero. Since the acoustic pressure is zero, then the velocity potential becomes zero automatically.

$$\nabla u = \tilde{v}, \Rightarrow \nabla u \cdot \tilde{n} = \tilde{v} \cdot \tilde{n} = 0 \Leftrightarrow \frac{\partial u}{\partial n} = 0, \Rightarrow \frac{\partial u}{\partial z} = 0 . \qquad (3.51)$$

The equation (3.40) in terms of finite difference becomes:

$$\frac{\partial u}{\partial z} \cong \left(\frac{u_{i,j+1} - u_{i,j-1}}{2h_z} \right) = 0 \Rightarrow u_{i,j+1} - u_{i,j-1} = 0 . \qquad (3.52)$$

Substituting $u_{i,j+1}$ for $u_{i,j-1}$ in equation (3.39) leads to

$$a_i u_{i-1,j} + g u_{i,j} + c_i u_{i+1,j} + (b+d) u_{i,j+1} = 0 . \qquad (3.53)$$

The equation (3.53) is used to model the boundary condition on the open bottom surface of the cylinder.

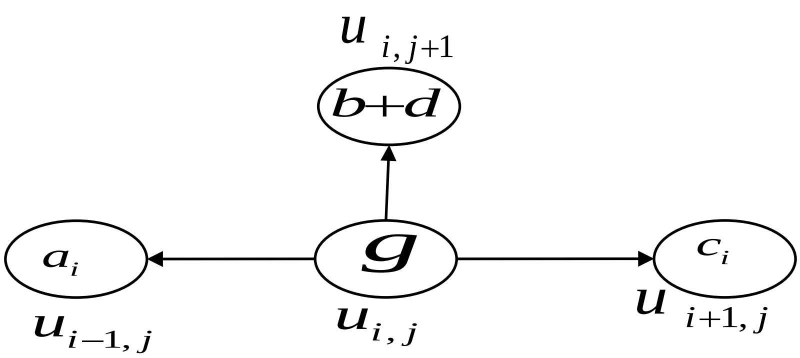

Figure 3. 7: Computational Molecule along the Bottom Cylinder Boundary

3.9 Forming the System of Linear Equations

The MATLAB Mathematical software developed (Appendix B) was used to generate the system of linear equation;

$$[A][u] = [f] \tag{3.54}$$

where $[u]$ is a 320 by 1 column vector whose components are containing the pressure at interior and boundary points, $[A]$ is the 320 by 320 matrix whose elements are the coefficients multiplying values of the pressure in the corresponding Helmholtz difference equation (3.39). The right hand side is the 320 by 1 column vector $[f]$ containing the specified values of pressure at the boundary of the domain. The system of equation (3.54) is solved by an iterative method using the developed MATLAB codes (Appendix B).

CHAPTER FOUR

MODEL SIMULATIONS

4.1 Analysis of Model Results

The finite difference method was used to approximate the solution of the Helmholtz equation (3.31) subject to the specified boundary conditions. The solution was a set of approximate values of the acoustic modal characteristics (natural frequencies, eigenmode shapes, etc.) of an open ended cylinder. MATLAB codes were written to determine the natural frequencies and the eigenmode shapes (see appendix B).

Simulation of the model was done using different open ended cylinder lengths and radii. Resonance for a chosen frequency, say middle C (260 Hz) was simulated by varying the value of the length in the MATLAB codes and fixing several chosen radius (0.02m, 0.04m and 0.05m). The model results exact half cycle eigenmode shape only at optimal resonant length as can be seen in Figure. 4.1. Any tiny variations in the length of the cylinder in MATLAB codes near the optimal resonant length change the eigenmode shape for producing resonance (Table 4.1). A similar test carried out by varying the radius of the cylinder from MATLAB codes, where the model results exact half cycle eigenmode shape as it can be seen in Figure 4.5 and 4.6 unless the radius is unreasonably large (Figure 4.7).

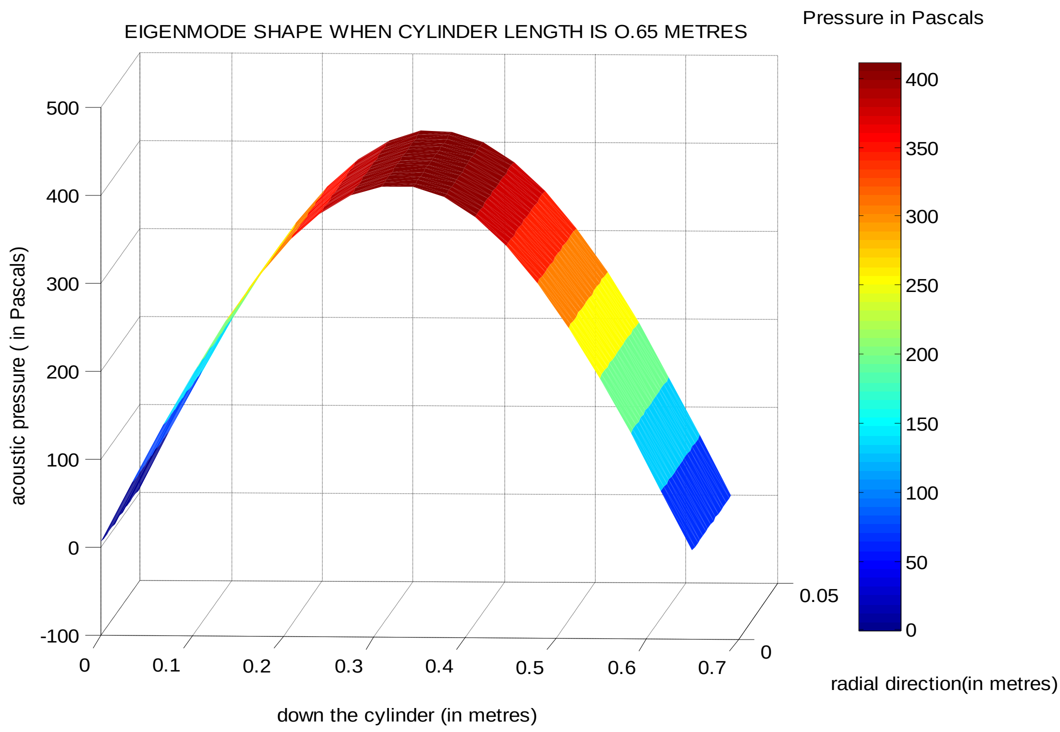

Figure 4.1: Eigenmode Shape at Optimal Resonant Length (Half-Cycle)

Table 4.1: Summary of the Model Data Results

	MODEL RESULTS							
RADIUS (M)	0.04	0.05	0.04	0.05	0.04	0.05	0.04	0.05
LENGTH(M)	0.55		0.60		0.65		0.70	
NATURAL FREQUENCY(HZ)	260		260		260		260	
EIGENMODE SHAPE	NOT A HALF-CYCLE		NOT A HALF-CYCLE		HALF-CYCLE		NOT A HALF-CYCLE	

4.2 Analysis of Experimental Results

4.2.1 PVC-Cylinders

PVC cylinders were used for the resonators because they are light, inexpensive, and easily available. Two open ended PVC-cylinders of different radii were used to test the validity of the model. A vernier caliper and meter ruler were used to obtain the dimensions (radius and length) of the two PVC- cylinders.

4.2.2 Experimental Set Up

In generally, resonance in an open ended cylinder is difficult to demonstrate using a tuning fork practice due to the instruments' low-intensity resonance. To overcome this problem, a speaker driven by a signal generator was used. The set-up used demonstrated resonance for a chosen frequency (middle C of 260 Hz) by changing the dimensions (length and radius) of the open ended cylinder. Two PVC cylinders of different size were chosen such that one could slide inside the other so as to demonstrate their natural frequency.

4.2.3 The Instruments Used in Experiment to Demonstrate Resonance in Cylinders.

In order to demonstrate the resonance in open ended cylinder (Figure 4.1) the instruments below were used.

(i) Signal generator (0.09Hz-110KHz),

(ii) The Oscilloscope (1Hz-25Hz),

(iii) The loud speaker of 6 inch producing full audible range of 20 Hz to20 KHz,

(iv) The small speaker (sensor) of a diameter of 4 inches with a range of 0 Hz to 20 KHz, used as a microphone,

(v) The pre-amplifier (40db) ,

(vi) Two sets of PVC cylinders of different radius were used.

Figure 4.2: Photograph of an Experimental Set up (UDSM).

Table 4.2: Summary of the Experimental Data Results from Appendix A.

	EXPERIMENTAL RESULTS			
NATURAL FREQUENY (HZ)	RADIUS (M)	LENGTH		**AMPLITUDE (VOLTS)**
		LENGTH	EFFECTIVE LENGTH (Le) **Le=L+(0.62*2r)**	
260	0.04	0.55	0.5996	0.76
		0.59	***0.63***	**0.8**
		0.66	0.70	0.63
260	0.05	0.55	0.61	0.84
		0.57	***0.63***	**0.9**
		0.66	0.71	0.46

4.3 Model Validation

For the purpose of validating the model, simulations were done to compare the model results and the laboratory experimental work results as summarized in Table 4.3

Table 4.3: Model versus Laboratory Experimental Data Results

RADIUS(M)	r=0.04			r=0.05		
RESULTS	**MODEL**	EXPERIMENTAL		**MODEL**	EXPERIMENTAL	
RESONANT LENGTH(M)	L	L	EFFECTIVE LENGTH(Le) Le=L+0.62*2r	L	L	EFFECTIVE LENGTH(Le) Le=L2+0.62*2r
	0.65	0.59	0.63	0.65	0.57	0.63
% error between model and experimental results	2%			2%		

4.4 Sensitivity Analysis

In this section the general relation between the setting down of the fundamental natural frequency (260 Hz) with respect to the dimensions parameters (length and radius) of an open ended cylinder as determined by using the mathematical model developed and experimental work performed, was analysed (Table 4.1).

4.4.1 Effects of Small Changes in the Length to the Natural Frequency of the Resonator

Small changes in length from the optimal resonant length cause the changes on the natural frequency. This effect is revealed through the model results (Figure 4.2 and 4.4), which does not give an exact half cycle eigenmode shape. Therefore, effects of a small changes in the length of the cylinder resonator has a direct impact on its natural frequency, and that, in fact, the natural frequency is very sensitive to small variations about its optimal resonant length.

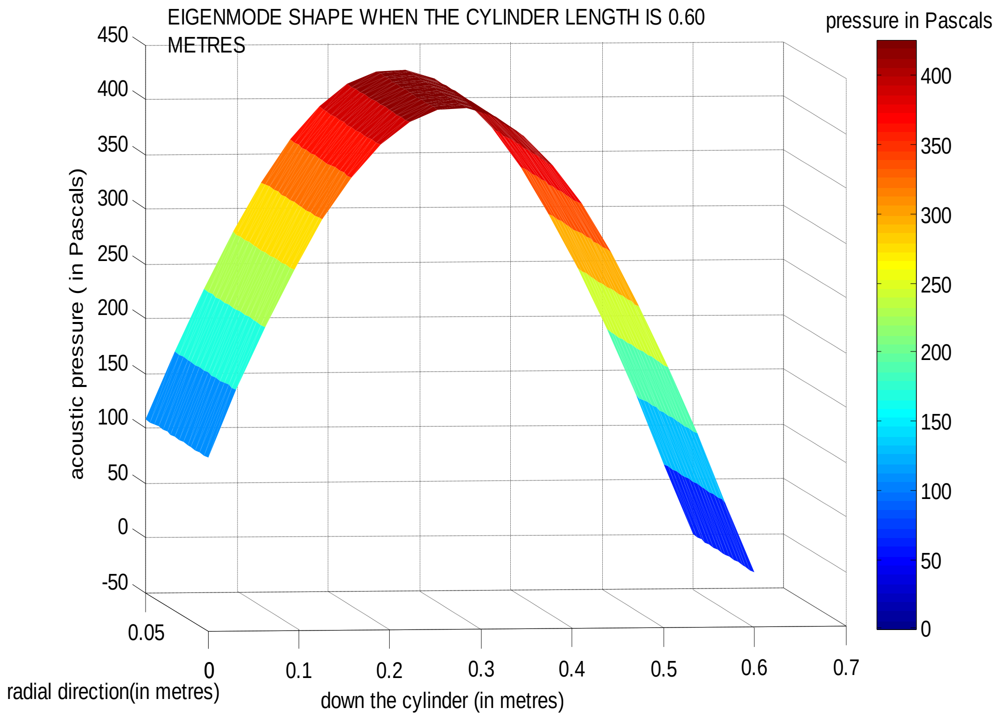

Figure 4.3: Eigenmode Shape when the Cylinder Length is 0.60 m

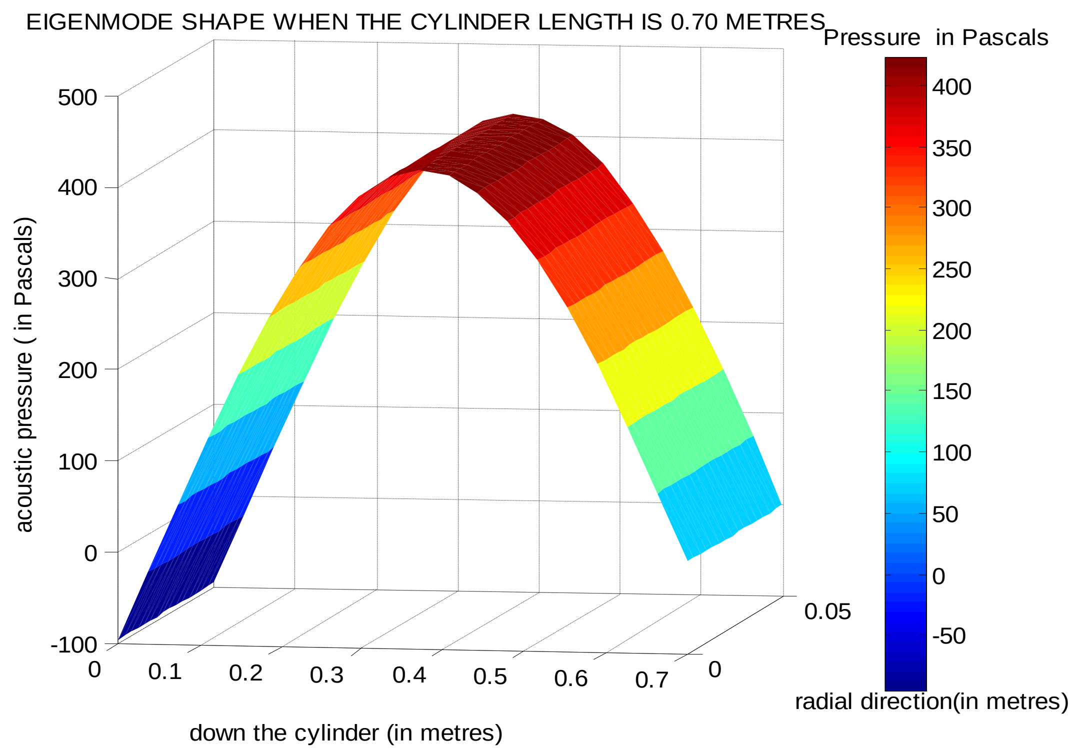

Figure 4.4: Eigenmode Shape When the Cylinder Length is 0.70 m

4.4.2 Effects of Small Changes in the Radius to the Natural Frequency of the Resonator

The model results reveal that, using different radii (say, 0.02, 0.04 and 0.05), unlike in the case of the length, the natural frequency of the cylinder resonator is not only independent from but also insensitive to variations in its radius (Figure 4.5 and 4.6), unless its radius is very large (Figure 4.7).

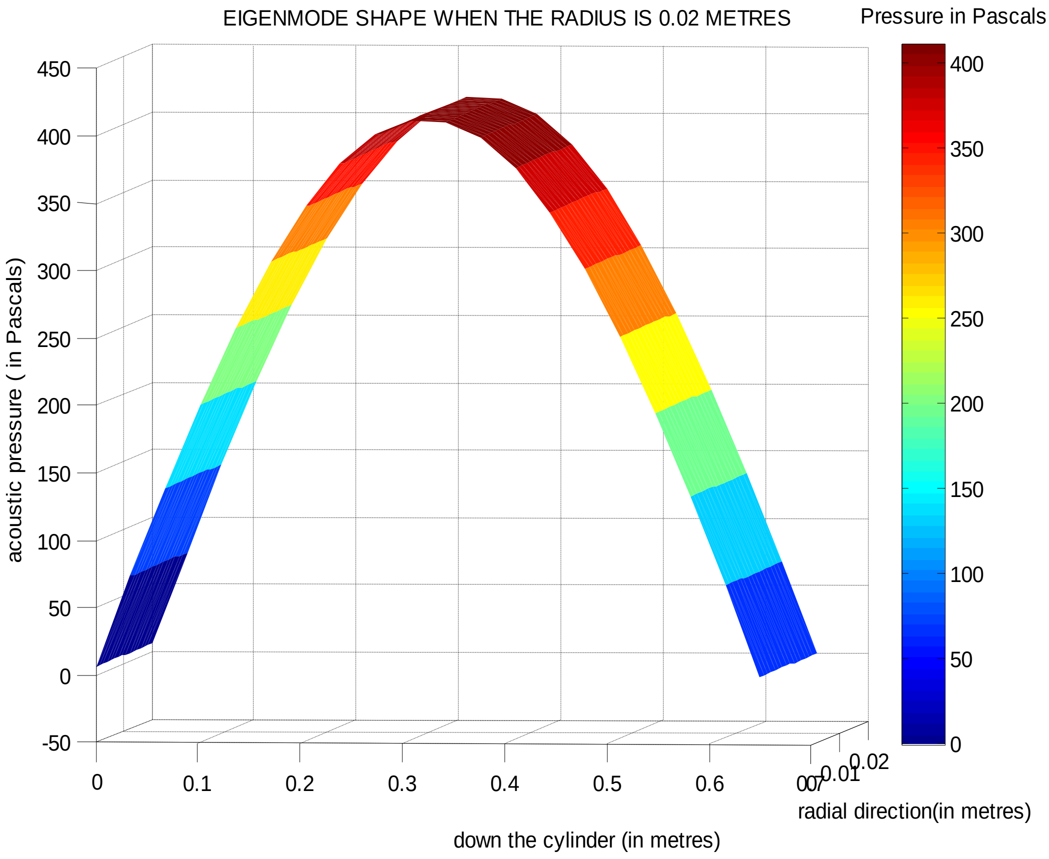

Figure 4.5: Eigenmode Shape When the Cylinder Radius is 0.02 m

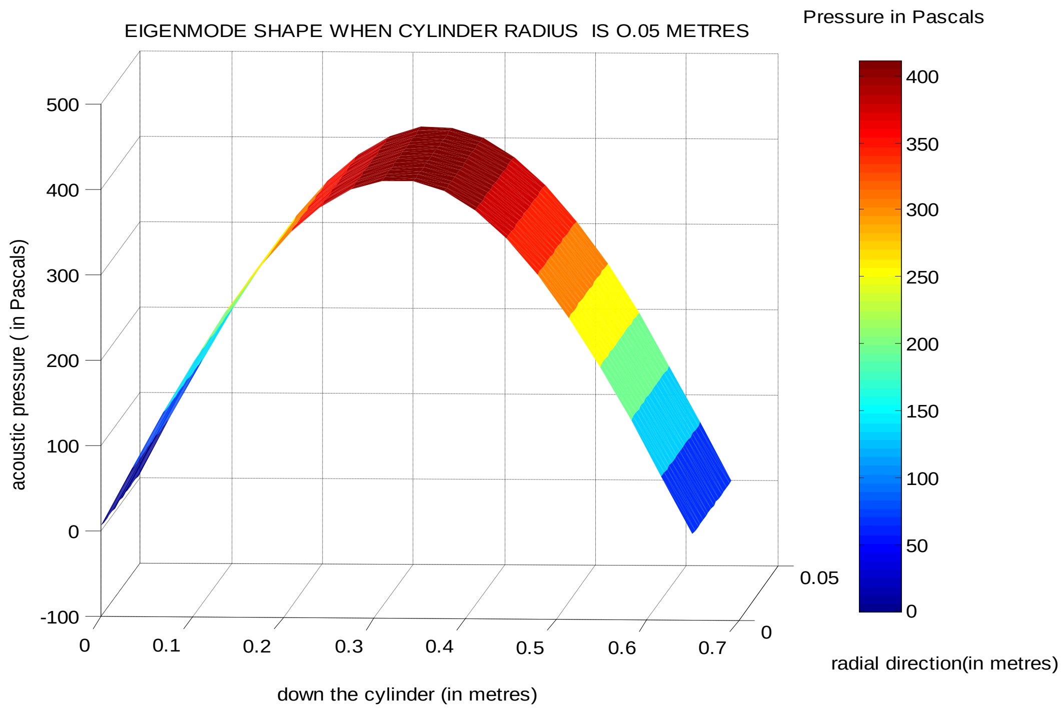

Figure 4.6: Eigenmode Shape When the Cylinder Radius is 0.05 m

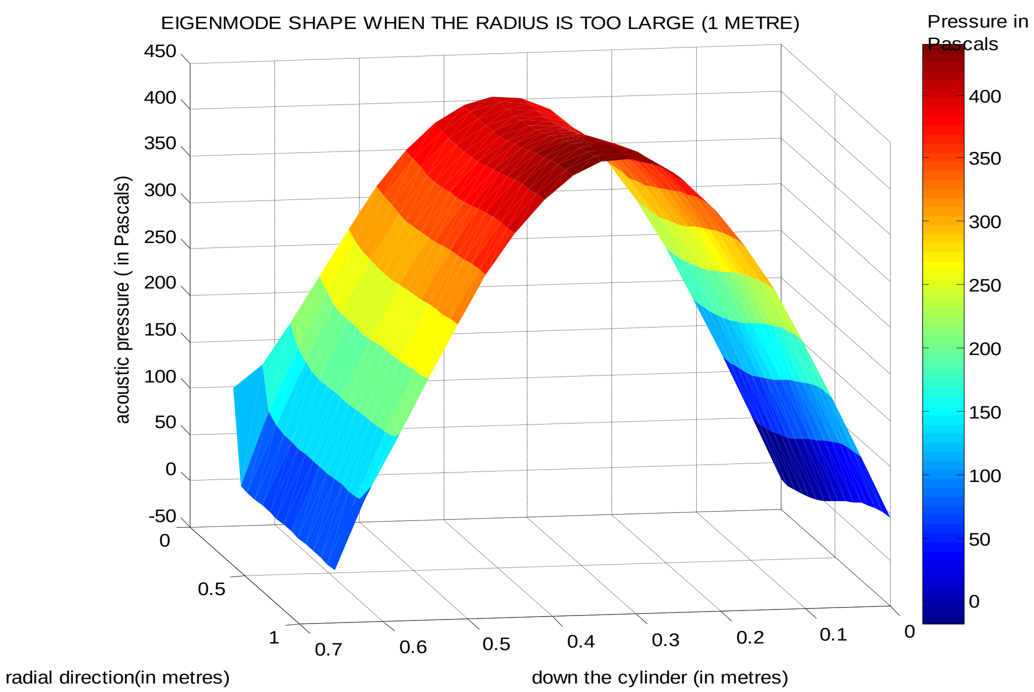

Figure 4.7: Eigenmode Shape When the Cylinder Radius is too Large.

CHAPTER FIVE

DISCUSSION, CONCLUSION, RECOMMENDATIONS AND FUTURE WORK

5.1 Discussion

This research set out to formulate a mathematical model for resonance in an open-ended cylinder. The aim was to find out the possible dependence or independence of the natural frequency of such a cylinder on its length and radius, and hence the possible application of the results in the design and manufacture of the Marimba musical instrument, whose resonators are cylinders of different lengths and radii.

The model turned out to be the well known two-dimensional time independent Helmholtz partial differential equation, extracted from the three-dimensional time dependent wave equation. Because of the geometry of the resonator, cylindrical coordinates were introduced and used to transform the model equation from its original cartesian coordinates to cylindrical coordinates. Neumann-type boundary conditions were imposed on the solution both at the top and bottom ends of the cylinder, as well as along the axis and surface wall of the cylinder. The finite difference method was applied to convert the continuous problem to a corresponding discrete problem and the resulting system of linear equations was solved using the MATLAB software.

The numerical results obtained from simulations of the model using cylinders of various lengths and radii establish beyond doubt that the length of the cylinder has a direct influence on its natural frequency, and that, in fact, the natural frequency is very sensitive to small variations about its optimal resonant length. On the other hand, similar experiments using different radii reveal that, unlike in the case of the length, the natural frequency of the cylinder is not only independent from but also insensitive to variations in its radius.

5.2 Conclusion

This research had set out to establish three hypotheses:

5.2.1 Hypothesis One

It was hypothesised that, the relationship between natural frequencies and the dimensions of an open-ended resonant cylinder can be modeled mathematically using suitable differential equations. The model has been formulated using the finite difference method (FDM) in chapter three equation (3.39). The model was used to determine the solution of the Helmholtz equation (3.31). The model was used to determine relationship between the natural frequencies and the dimensions (length and radius) of an open-ended resonant cylinder.

5.2.2 Hypothesis Two

It was hypothesised that, the natural frequencies of cylinders are related to the dimensions of the cylinder. The mathematical model developed was intended to model the resonance in an open ended cylinder. The model can predict the relationship between the natural frequencies and its length at a deviation of 2% (Table 4.3).

The results of the mathematical model (Figure 4.2, 4.3, and 4.4) and that of laboratory experimental work (Table 4.1) show that, there is a very close relationship between the natural frequencies of resonant cylinders and its length. This variation of the cylinder length affects the cylinder resonance. For example, at optimal resonant length of 0.65 meters (Figure 4.3), the eigenmode shape is half a cycle and moving away in either direction from this mark does not give an exact half cycle (Figure 4.2 and Figure 4.4).

Moreover, a variation of radius (Figure 4.5 and Figure 4.6) was not causing the sound pressure to vary until the radius was very large (Figure 4.7).

5.2.3 Hypothesis Three

It was hypothesised that, solutions of the mathematical model using suitable mathematical software are good approximations of the analytical solutions and are comparable to laboratory experimental results. The model results provide good approximations to the actual laboratory experimental results with a difference of 2% (Table 4.3).

5.3 Recommendations

Since it is revealed from both simulation of the model and validation of the model and confirms the researcher's hypotheses that, the length of the cylinder has a direct influence on its natural frequency, and that it is insensitive to variations in its radius then, it is recommended to Marimba manufacturers to take highest consideration on the factor of length in order to get the required resonant tone.

5.4 Future Work

The study involves mathematical modelling of resonance in open ended cylinder .The case of Marimba resonator. It can be extended by changing the numerical approach technique from using FD to FE.

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APPENDICES

Appendix A

Laboratory Experimental Data Results

LENGTH (L)	EFFECTIVE LENGTH(Le) **Le=L+(0.62*2r)**	AMPLITUDE (Volt)	EFFECTIVE LENGTH(Le) **Le=L+(0.62*2r)**		AMPLITUDE (Volt)
RADIUS	R=0.04		R=0.05		
0.4	0.4496	1.4*0.1	0.4	0.462	2.0*0.1
0.41	0.4596	2.2*0.1	0.41	0.472	2.2*0.1
0.42	0.4696	2.4*0.1	0.42	0.482	2.4*0.1
0.43	0.4796	2.8*0.1	0.43	0.492	2.6*0.1
0.44	0.4896	3.0*0.1	0.44	0.502	2.8*0.1
0.45	0.4996	3.4*0.1	0.45	0.512	3.2*0.1
0.46	0.5096	4.2*0.1	0.46	0.522	3.6*0.1
0.47	0.5196	4.4*0.1	0.47	0.532	3.8*0.1
0.48	0.5296	4.8*0.1	0.48	0.542	4.2*0.1
0.49	0.5396	5.*0.1	0.49	0.552	4.5*0.1
0.5	0.5496	5.2*0.1	0.5	0.562	5.2*0.2
0.51	0.5596	5.8*0.1	0.51	0.572	5.6*0.2
0.52	0.5696	3.1*0.2	0.52	0.582	3.4*0.2
0.53	0.5796	3.2*0.2	0.53	0.592	3.6*0.2
0.54	0.5896	3.6*0.2	0.54	0.602	3.8*0.2
0.55	0.5996	3.8*0.2	0.55	0.612	4.2*0.2
0.56	0.6096	3.9*0.2	0.56	0.622	4.4*0.2
0.57	0.6196	3.95*0.2	***0.57***	***0.632***	4.5*0.2
0.58	0.6296	3.95*0.2	0.58	0.642	4.4*0.2
0.59	***0.631***	4.0*0.2	0.59	0.652	4.3*0.2
0.6	0.6496	3.92*0.2	0.6	0.662	1.8*0.2
0.61	0.6596	3.95*0.2	0.61	0.672	3.8*0.2
0.62	0.6696	3.9*0.2	0.62	0.682	3.4*0.2
0.63	0.6796	3.7*0.2	0.63	0.692	3.2*0.2
0.64	0.6896	7.2*0.1	0.64	0.6896	5.6*0.1
0.65	0.6996	6.4*0.1	0.65	0.6996	4.8*0.1
0.66	0.7096	6.3*0.1	0.66	0.7096	4.6*0.1
0.67	0.7196	5.8*0.1	0.67	0.7196	4.2*0.1
0.68	0.7296	5.0*0.1	0.68	0.7296	3.8*0.1
0.69	0.7396	4.3*0.1	0.69	0.7396	3.4*0.1
0.7	0.7496	4.0*0.1	0.7	0.7496	3.2*0.1
0.71	0.7596	3.2*0.1	0.71	0.7596	2.8*0.1
0.72	0.7696	2.4*0.1	0.72	0.7696	2.2*0.1
0.73	0.7796	1.6*0.1	0.73	0.7796	1.8*0.1

Appendix B

MATLAB CODES

```
close all

clear

% PROGRAMME TO SOLVE HELMHOLTZ EQUATION

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% INPUT DATA

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%  Lr = the lengths of cylinder in the r direction

%  Lz=the lengths of cylinder in the z direction

%  nr = number of nodes in the r direction

%  nz = number of nodes in the z direction

%  hr= mesh width in the r-direction

%  hz= mesh width in the z-direction

%nn=Total number of non-zero nodes

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Enter simulation parameter

%nn = ; % Number of non-zero nodes

%fq =260; % Frequency at middle C in Hz

% rho = 1.21; % Density of air medium in kg/m^3

% K=Wave number

% cs= velocity of propagating  sound waves in m/s

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
Lz =0.65;
cs=340;
rho = 1.21;
fq =260;
Lr  =0.05;
k   = 2*pi*fq/cs;
w   = 2*pi*fq;
nr  =16;
nz  =20;
hr  = Lr/(nr-1); % introduced a bracket to correct an error
hz  = Lz/(nz-1);
nn  = nr*nz;
```

%PROGRAMMING THE FINITE DIFFERENCE APPROXIMATION

FORMING THE MATRIX A AND VECTOR f WHICH GIVE THE SOLUTION FROM **Au = f**

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
A = zeros(nn);
f = zeros(nn,1);
% SETTING THE FD EQUATIONS FOR THE INNER NODES
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% west of p ,W=P-1;east of p ,E=P+1;south of p, S=P-nr %north of p,N=P+nr;
b = 1/(hz*hz);
d = 1/(hz*hz);
```

```
g = k*k-2/(hr*hr)-2/(hz*hz);
for   i = 2:nz-1    % i gives columns i.e. z variation
for   j = 2:nr-1    % j for rows i.e. radially
          P  = (i-1)*nr+j;
          rp = (j-1)*hr;                % tidied up by forming r at P.
 a  = (1/(hr*hr))-1/(2*hr*rp);
 c  = (1/(hr*hr))+1/(2*hr*rp);
          A(P,P)    = g;
          A(P,P-1)  = a;
          A(P,P+1)  = c;
          A(P,P-nr) = b;
          A(P,P+nr) = d;
  end
end
```

%%

% SETTING IN THE BOUNDARY CONDITIONS

%%

THE BOUNDARY CONDITION TO MODEL THE TOP SURFACE OF A CYLINDER INCLUDING THE AXIS AND OUTER SURFACE, ASSUMED THE ARBITRARYSOUND SOURCE STRENGTH OF ONE.

```
b = 1/(hz*hz);
d = 1/(hz*hz);
g = k*k-2/(hr*hr)-2/(hz*hz);
```

```
for  jt = (nz-1)*nr+2:nz*nr-1;    % j for rows i.e. radially
      P  = jt;
      rp = (jt-1)*hr;     % tidied up by forming r at P.
      a  = (1/(hr*hr))-1/(2*hr*rp);
      c  = (1/(hr*hr))+1/(2*hr*rp);
      A(P,P)    = g;
      A(P,P-1)  = a;
      A(P,P+1)  = c;
      A(P,P-nr) = d+b;
      f(P)=2*d*hz*v0;
  end
```

```
%  MODELLING THE LEFT HAND CORNER (CENTRE AXIS)
d  = 1/(hz*hz);
gm = k*k-4/(hr*hr)-2/(hz*hz);
cm = 4*(1/(hr*hr));
P  = (nz-1)*nr+1;
A(P,P)    = gm;
A(P,P+1)  = cm;
A(P,P-nr) = 2*b;
v0=1;
f(P)=2*v0*hz/(hz*hz);
```

```
%  MODELLING THE RIGHT HAND CORNER (OUTSIDE) CYLINDER
 b  = 1/(hz*hz);
```

```
gm = k*k-2/(hr*hr)-2/(hz*hz);
cm = 2*(1/(hr*hr));
     P  = nr*nz;
     A(P,P)    = gm;
     A(P,P-1)  = cm;
     A(P,P-nr) = 2*b;
     f(P)=2*v0*hz/(hz*hz);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

THE BOUNDARY CONDITION THAT MODELS THE INNER NODES OF THE CYLINDER WALL

```
 b  = 1/(hz*hz);
 d  = 1/(hz*hz);
 g  = k*k-2/(hr*hr)-2/(hz*hz);
 am = 2*(1/(hr*hr));
for it=(2*nr):nr:nr*nz-nr
    A(it,it)    = g;
    A(it,it-1)  = am;
    A(it,it-nr) = b;
    A(it,it+nr) = d;
    f(it)       = 0;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

THE BOUNDARY CONDITION THAT MODEL THE SYMMETRY AXIS OF THE CYLINDER AT THE INNER NODES (NOT TOP OR BOTTOM NODES)

```
b= 1/(hz*hz);
d= 1/(hz*hz);
gm = k*k-4/(hr*hr)-2/(hz*hz);
cm = 4*(1/(hr*hr));
 for
     it= nr+1:nr:(nz-2)*nr+1
     P  = it;
     rp = (j-1)*hr;  % tidied up by forming r at P.
     A(P,P)    = gm;
     A(P,P+1)  = cm;
     A(P,P-nr) = b;
     A(P,P+nr) = d;
  end
```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

THE BOUNDARY CONDITION THAT MODEL THE BOTTOM OF THE %CYLINDER WHEN OPEN IS THAT, THE ACOUSTIC CPRESSURE IS ZERO, %AND SO THE VELOCITY POTENTIAL IS ZERO USING AN ARTIFICIAL %NODE AND THE COMPUTATIONAL MOLECULE.

```
for it=1:nr
    A(it,:)=0;
    A(it,it)=1;
    f(it)=0;
```

```
            end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%
% MODELLING THE LEFT HAND CORNER (CENTRE AXIS)
        d  = 1/(hz*hz);
        gm = k*k-4/(hr*hr)-2/(hz*hz);
        cm = 4*(1/(hr*hr));
         P  = 1;
        A(P,P)    = gm;
        A(P,P+1)  = cm;
        A(P,P+nr) = 2*d;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% MODELLING THE RIGHT HAND CORNER (OUTSIDE) CYLINDER
        d  = 1/(hz*hz);
        gm = k*k-2/(hr*hr)-2/(hz*hz);
        cm = 2*(1/(hr*hr));
        P  = nr;
        A(P,P)    = gm;
        A(P,P-1)  = cm;
        A(P,P+nr) = 2*d;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% THE BOUNDARY CONDITION THAT MODELS THE INNER NODES OF
 % THE CYLINDER WALL
        b  = 1/(hz*hz);
```

```
            d  = 1/(hz*hz);
            g  = k*k-2/(hr*hr)-2/(hz*hz);
            am = 2*(1/(hr*hr));
            for it=(2*nr):nr:nr*nz-nr
            A(it,it)    = g;
            A(it,it-1)  = am;
            A(it,it-nr) = b;
            A(it,it+nr) = d;
  f(it)       = 0;
 end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
u = A\f;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%PLOTTING SECTION
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ur = zeros(nn,1);
for it = 1:nn
ur(it) = u(nn+1-it); % re-ordering to start numbering from the top
end
r = 0:hr:Lr; % setting the radial axis starting at the centre
z = 0:hz:Lz; % setting the z-axis starting at the top
DM = rho*w*reshape(ur,nr,nz);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
legend off
```

```
figure(1)
mesh(z,r,DM)
ylabel('radial direction(Lr)')
xlabel('down the cylinder-Lz')
zlabel('acoustic pressure-DM')
shading interp
colormap(jet)
view(3)
title(['Finite difference to approximation Helmholtz equation '])
colorbar
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure(2)
contour(z,r,DM,50)
ylabel('radial direction-Lr')
xlabel('down the cylinder-Lz')
zlabel('acoustic pressure-DM')
title(['Contours of finite difference approximation with h = ,num2str(h)'])
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%
figure(3)
surf(z,r,DM)
ylabel('radial direction-Lr')
xlabel('down the cylinder-Lz')
zlabel('acoustic pressure-DM')
```

```
shading interp

colormap(jet)

view(3)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% END OF PROGRAMME
```